DEPARTMENT OF PHYSICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Master's Thesis in Physics (Nuclear, Particle and Astrophysics)

New Quantum Effects on Neutrino Parameters

Lukas Treuer

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Neue Quanteneffekte auf Neutrinoparameter

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I confirm that this master's thesis in physics (nuclear, particle and astrophysics) is my own work and I have documented all sources and material used.

Garching bei München, 23.03.2023 Lukas Treuer

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Abstract

Despite the longstanding assumption that neutrinos are massless, oscillation experiments have shown that they are, in fact, massive, and furthermore mix significantly. Constraints from other observations additionally indicate that these masses are very small compared to others in the Standard Model. A possible explanation is provided by the seesaw mechanism, wherein heavy intermediate particles suppress the masses of the Standard Model neutrinos. On the other hand, the unique dimension-five Weinberg-Operator provides a model-independent description of Majorana neutrino masses. Restrictions on its structure, and thus the mixing angles, are often induced via flavor symmetries. Furthermore, radiative running effects can significantly impact neutrino phenomenology due to the different scales involved in mass generation, neutrino creation, and detection mechanisms.

In this work, we combine the above-mentioned paradigms, and we consider the renormalization group equations (RGEs) of the Weinberg-Operator in flavor-nonuniversal gauge theories. We find that in such models, the new gauge bosons induce novel terms in the beta-function of the neutrino mass matrix at the one-loop level. These terms can raise the rank of the mass matrix at the one-loop level, and generate up to three neutrino masses via RGE running only. Using a series of loop-topological and symmetry arguments, we derive the most general RGEs for the Weinberg-Operator and mass eigenvalues, and discuss their origin. We furthermore prove that the previously known formula to calculate the one-loop beta-function from renormalization constants holds in full generality. We also provide formulae that verify straightforwardly whether the new terms discovered in this work appear in any theory of interest. Lastly, we develop an explicit gauged $U(1)_{L_\mu - L_\tau}$ model with six new scalars in a Type-I seesaw mechanism involving three right-handed neutrinos. We calculate the complete RGEs for the right-handed neutrinos and the effective neutrino mass matrix therein, and discuss issues that arise in this type of paradigm.

Contents

PART

I

Introduction, Neutrino Physics, and Renormalization Theory

In the first part of this work, we will introduce the general field and specific topic of this work. Furthermore, we will review neutrino physics, covering why neutrino masses are a necessary extension of the Standard Model, neutrino oscillations, models for the generation of neutrino masses, and the effective description thereof. Lastly, we will discuss renormalization, renormalization group equations, and in particular the Standard Model renormalization of the effective operator describing neutrino masses. In the second part of this work, we will then apply the principles discussed here to a more general framework, and concrete extensions of the Standard Model.

CHAPTER

I-1 Introduction

Neutrinos are described as massless in the Standard Model of Particle Physics, but nevertheless, oscillation experiments have shown that they are, in fact, massive [1, 2]. In particular, these experiments require at least two nonzero masses, and constrain the mixing angles between the flavor and mass eigenbases—for recent measurements see, e.g., [3, 4]. Further constraints come, for instance, from beta decays and cosmological observations [5, 6], which provide limits on an effective mass of the electron neutrino, and the sum of neutrino masses, respectively. From these experimental results, it is clear that neutrino masses are significantly smaller than those of the charged leptons and other elementary fermions.

While the existence of neutrino masses is well-known at this point, it is not clear how these arise. In fact, due to their electric neutrality, weak coupling, and small mass, even the fundamental nature of neutrinos as Dirac or Majorana particles is yet unknown. In the Dirac case, neutrinos and anti-neutrinos are distinct, whereas Majorana neutrinos would be their own antiparticles. The mechanism generating the neutrinos' masses may thus further shine light on this issue, as it may restrict the nature of neutrinos. Experimental evidence of the underlying nature of neutrinos could, for instance, be provided by neutrinoless double beta decay. Therein, particular nuclei undergo two beta decays simultaneously without the emission of neutrinos, which can only occur if neutrinos are Majorana particles. This is because the two emerging antineutrinos need to annihilate with each other, which requires particle and antiparticle be the same. Thus, observation of neutrinoless double beta decay would prove their nature as Majorana particles. Non-observation, while not disproving this hypothesis, provides a limit on the effective mass of electron neutrinos, which is related to the rate of the process.

Furthermore, a reasonable creation mechanism of neutrino masses should explain their comparative smallness; a particularly well-known example is the seesaw mechanism. In this class of models, the smallness of neutrino masses is explained by very heavy intermediate particles that effectively suppress the neutrino's masses. In particular, one may introduce heavy, right-handed neutrinos in so-called Type-I seesaw models. The masses of the Standard Model neutrinos are then proportional to the squared vacuum expectation value of the Higgs field and the inverse mass of the right-handed neutrinos. Thus, if these right-handed neutrinos are very heavy, the resulting mass of the Standard Model neutrinos is very light. In particular, this mechanism leads to Majorana masses for the neutrinos, which can be described in an effective field theory by the so-called Weinberg-Operator [7]. The effective description via this unique dimension-five operator has the advantage that it can be applied in a model independent way. It is obtained from the full high-energy theory by integrating out the heavy intermediate fields, which yields an effective contact interaction between the Higgs-doublet and the lepton-doublet. Thus, after electroweak symmetry breaking, this effective operator gives masses to the neutrinos.

As opposed to the quark sector, the mixing angles in the neutrino sector appear to be large, which led to proposals of flavor symmetries imposing particular structures on the neutrino mass matrix—see, e.g., [8] for discrete symmetries. Even though such symmetries may be lightly broken, they provide frameworks to explain the large mixing angles, and potentially mass splittings. Other attempts extend the Standard Model gauge group by additional flavor gauge groups, most notably $U(1)$. In particular, gauging the difference of two lepton flavors is investigated, as it is free of gauge anomalies even without extending the fermion sector [9]. However, these impose stringent constraints on the structure of the mass matrix, such that the simplest models are already excluded by oscillation data [10]. Nevertheless, gauging the difference of muon and tau family numbers in particular, is interesting as it may also explain other phenomena, such as the anomalous magnetic moment of the muon or dark matter [11, 12]. In such scenarios, the symmetry is often only broken slightly, with the mass of the emerging massive gauge boson not significantly higher than the weak scale.

In addition to the tree-level description of neutrino masses, radiative corrections and running effects have also been considered [13, 14, 15, 16, 17, 18, 19]. The renormalization group running of neutrino parameters may give significant contributions at low energies, and furthermore allows us to evolve the mass matrix down from higher energy scales, potentially predicting low-energy phenomenology. Furthermore, as such radiative corrections provide a way to describe energy dependence of neutrino parameters, they may be detectable at oscillation experiments, and potentially explain experimental anomalies [20, 21].

In this work, we combine the previously mentioned aspects, and investigate the effects of flavor gauge theories on the renormalization, and running of neutrino parameters. In particular, we consider frameworks where the flavor symmetry is badly broken, washing out the original mass matrix structure. Therefore, the flavor gauge bosons considered in this work are much heavier than the ones used to explain, e.g., the anomalous magnetic moment of the muon. Independently of these scales, we derive the most general renormalization group equations for the effective dimension five neutrino mass operator at the one-loop level. Through our considerations, we find that the renormalization group equations contain additional terms in the presence of flavor-nonuniversal gauge interactions, and that these novel terms have significant impact on neutrino phenomenology. We provide general formulae to calculate these new terms in any extension of the Standard Model, and discuss the general framework in-depth. We furthermore prove general properties of the constituents of the renormalization group equations, and how to calculate them at the one-loop level. Lastly, we provide an explicit example of a complete model exhibiting the new quantum effect discovered in this work.

CHAPTER

I-2 Neutrino Physics

Let us begin this work by introducing the field of neutrino physics. We will cover various salient aspects, such as oscillations and experimental evidence for neutrino masses, models for neutrino masses, and the effective description thereof.

We will base this chapter on [22, 23, 24, 25] and further sources as referenced.

I-2.1 Motivation and Evidence for Neutrino Masses

Before discussing evidence for neutrino masses, let us review why they are assumed to be massless within the Standard Model (SM). Neutrinos are neutral particles—in fact, the only neutral fermions in the Standard Model. Therefore, we would like to define some quantum number, some charge, to distinguish neutrinos from anti-neutrinos. A reasonable choice for such a charge is lepton number, counting the number of charged leptons—electrons, muons, and tauons—and their corresponding neutrino partners. Recall that in the SM, the left-handed charged leptons and neutrinos are described as two components of a left-handed lepton-doublet of $SU(2)_L$. The right-handed charged leptons are singlets under $SU(2)_L$, and we do not know of any right-handed neutrinos; within the SM only left-handed ones exist. This is motivated by the maximal parity violation of the weak interactions, and the fact that as far as we know, the neutrinos only interact weakly. Now, if lepton number is conserved, we can use it to distinguish neutrinos carrying lepton number +1, and anti-neutrinos carrying lepton number -1. In this case, we speak of *Dirac neutrinos*—neutrinos and anti-neutrinos are distinct particles. If not, however, and neutrinos are their own anti-particles, we speak of *Majorana neutrinos*. In particular, Majorana neutrinos would fulfill the condition

$$\nu^C = \nu\,, \qquad \nu^C \equiv C\,\overline{\nu}^T = C\,\gamma^0\,\nu^*\,, \tag{I-2.1}$$

where the charge conjugation operator is given by $C = i\,\gamma^2\,\gamma^0$. This is called the Majorana condition, and cannot be fulfilled if the particles carry a charge—the transformation of ν would be opposite, and thus inconsistent with that of ν^*. The mass terms for Dirac and Majorana neutrinos in terms of left- and right-handed fields are of the form

$$-\mathcal{L}_{\text{Dirac mass}} \sim \overline{\nu_L}\, m\, \nu_R + \overline{\nu_R}\, m\, \nu_L\,, \qquad \text{and} \tag{I-2.2}$$

$$-\mathcal{L}_{\text{Majorana mass}} \sim \overline{\nu_L^C}\, m_{\nu_L}\, \nu_L\ \big(+\overline{\nu_R^C}\, m_{\nu_R}\, \nu_R\big)\,. \tag{I-2.3}$$

Let us now consider the viability of these two options. Following observations of, e.g., electron energies in beta decays of nuclei via the weak interaction, we know that neutrino masses would have to be small, meaning that the coupling between the left- and right-handed neutrinos in the Dirac case is small. Furthermore, right-handed neutrinos would be $SU(2)_L$ singlets, and as such, not interact via the weak interaction. However, as far as we know, neutrinos only interact via the weak force, meaning that the right-handed neutrinos would be extremely difficult to detect. Thus, as we have not observed right-handed neutrinos so far, they are not part of the SM. This then means that we cannot have a Dirac mass term, as a right-handed counterpart to the SM neutrinos would be necessary.

On the other hand, we find that lepton number is conserved in all observed interactions, as are the lepton flavor numbers—the number of leptons of the electron, muon, and tauon families. Therefore, we may also draw the conclusion that if neutrinos are Majorana particles, they do not have a lepton-number-breaking mass term. Together with the previous arguments, this leads to the SM assumption of *massless* neutrinos. Then, however, the following question arises:

I-2.1.1 Why Are We Interested in Neutrino Masses?

The answer to this question lies, in the solar neutrino problem. This problem refers to the long-standing mismatch between the number of electron neutrinos expected following models for the nuclear interaction in the sun, and the number of detected electron neutrinos on earth. This meant that either the standard solar model was wrong, or neutrinos went missing along their way to earth. Let us first consider how the neutrinos are created. One notable source of neutrinos in the sun is the pp chain, a sequence of nuclear fusion reactions that starts with the fusion of two protons to a deuteron, and of a proton and deuteron to helium-3,

$$p + p \longrightarrow d + e^+ + \nu_e \qquad\qquad p + d \longrightarrow {}^3\text{He} + \gamma\,, \tag{I-2.4}$$

where p is the proton, d the deuteron, and ^{3}He helium-3. We can understand the final products of eq. (I-2.4) from the fact that we need to conserve total charge, giving us a positron, and the neutrino ensuring conservation of lepton number and electron family number. From helium-3 further elements can be created, among others boron, which can then undergo beta decay,

$$^8B \longrightarrow {}^8Be + e^+ + \nu_e\,. \tag{I-2.5}$$

From this beta decay, we see that we obtain beryllium, and yet again another neutrino. Via these and other reactions, electron neutrinos are created in the sun, which we can then detect in detectors on earth. Note that the neutrinos coming from different reactions also bear different energies, and depending on the respective reaction rate, different fluxes. In fig. (I-2.1) we show the neutrino flux from different interactions, as a function of the neutrinos' energy.

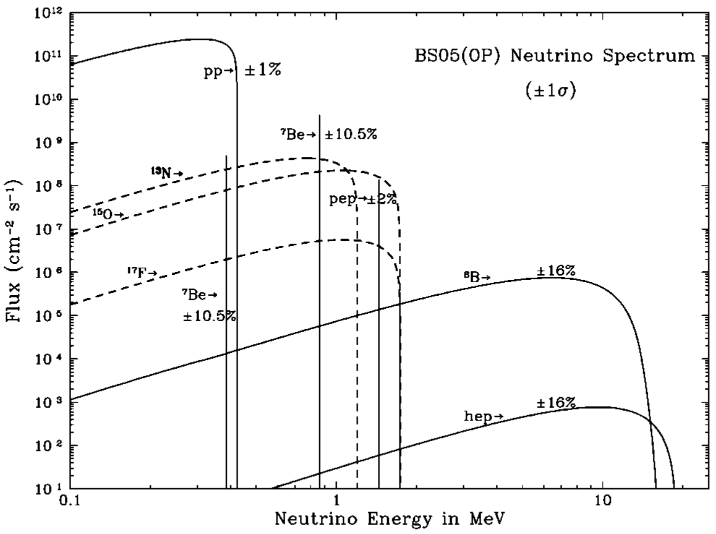


Figure I-2.1.: Flux of solar neutrinos as a function of their energy, depending on their creation reaction [26]

The definitive resolution of this impasse came by way of the Sudbury Neutrino Observatory (SNO), proving that there were non-electron neutrinos coming from the sun, while the total neutrino flux agreed with the standard solar model [1]. We show the plot summarizing their findings in fig. (I-2.2).

Another avenue to measure neutrino oscillations is the via atmospheric neutrinos. These are created in the atmosphere, where cosmic ray scatterings create, e.g., pions. These subsequently decay weakly into muons and muon anti-neutrinos. The muons then decay further into electrons and electron anti-neutrinos. Note that this is the chain for negatively charged pions, in the case of positively pions, anti-muons and muon neutrinos, etc. are created in the decay. Naively, we would expect the azimuthal angle from which we detect the neutrinos not to have an impact on the measured flux. The reasons for this are that since neutrinos barely interact with matter, we can neglect the loss incurred by neutrinos passing through the earth; furthermore, the cosmic ray flux is isotropic, so this should not impact the directionality of detection either.

However, the Super-Kamiokande Cherenkov experiment found only an agreement with this expectation for the electron neutrino, whereas the muon neutrino flux exhibited an angle dependence [2]. We show their results in fig. (I-2.3), where the results for μ-like neutrinos with energies larger than 0.4 GeV in the sub-GeV range are particularly illuminating. Namely, we see that the neutrino flux is close to the expected value without oscillations for large $\cos\theta$—corresponding to neutrinos coming from above the experiment. However, for $\cos\theta < 0$—corresponding to particles coming from below the experiment—the measurements

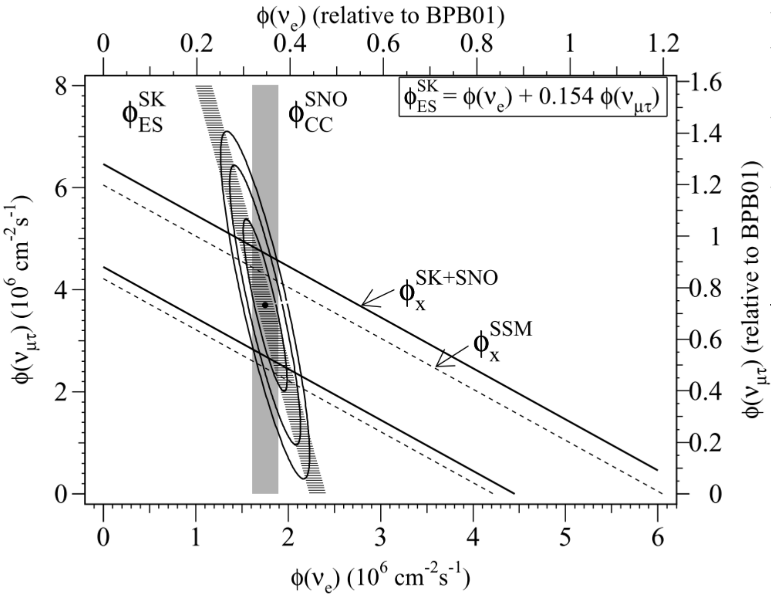


Figure I-2.2.: Flux of electron neutrinos, and combined muon and tau neutrinos from different detection channels for solar neutrinos in the SNO experiment, partially combined with data from the Super-Kamiokande experiment; taken from [1]

are very different from the expectation without oscillations. And in fact, the smaller $\cos\theta$, the bigger the discrepancy. Therefore, we know that this effect depends on the distance traveled between the neutrino creation in the atmosphere, and the detection in the experiment. The reason for is that the longer the neutrino travels before detection, the greater the probability it underwent flavor oscillation. Another aspect we see from the Super-Kamiokande data, is that the oscillations depend on the energy of the neutrinos as well, since the effects are not identical, e.g., for above and below $0.4\,\text{GeV}$ energies. Furthermore, since the electron neutrino flux is consistent with the expectation for absent oscillations, we know that electron neutrinos do not oscillate noticeably in the distance and energy range relevant for atmospheric neutrinos.

There are further experimental verifications of neutrino oscillation, such as accelerator or reactor experiments. While we will not cover those here, we see that evidence for the presence of oscillations is abundant. Let us therefore see how the observation of neutrino oscillations necessitates their nonzero masses.

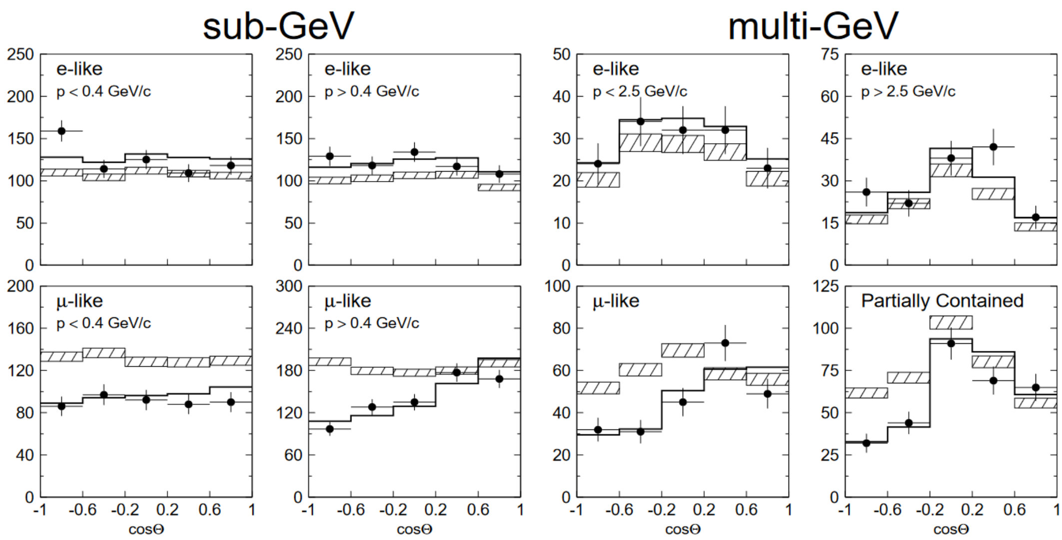


Figure I-2.3.: Flux of atmospheric electron and muon neutrinos of different energies as a function of the cosine of the azimuthal angle θ; $\cos\theta < 0$ for up-, and $\cos\theta > 0$ for downward going neutrinos; hatched regions are the expected values without oscillation, the solid line is with oscillations between ν_μ and ν_τ, the points are the experimental measurements; taken from [2]

I-2.1.2 Neutrino Oscillations

To describe neutrino oscillations, let us assume we have neutrinos of a flavor a, ν_a, that is created at some source—e.g., the sun or the atmosphere. We are then interested in the probability of detecting a neutrino of flavor b, ν_b, at some later time and after ν_a has traveled some distance. Therefore, we are interested in the matrix element

$$\langle \nu_b | \nu_a(\vec{x},\, t) \rangle = \left\langle \nu_b \,\middle|\, \exp\left(-\, i\, \hat{H}\, t + \hat{\vec{P}} \cdot \vec{x} \right) \middle|\, \nu_a \right\rangle \tag{I-2.6}$$

where $\hat{H}$ is the Hamiltonian and $\hat{\vec{P}}$ the momentum operator, responsible for time evolution and translation, respectively. We can now insert a complete set of energy eigenstates ν_i in eq. (I-2.6),

$$\langle \nu_b | \nu_a(\vec{x},\, t) \rangle = \sum_i \sum_\sigma \int \mathrm{d}^3\vec{k}\, e^{-i\, E_i(k)\, t + i\, \vec{k}\cdot\vec{x}}\, \langle \nu_b | \nu_i(\vec{k}, \sigma) \rangle\, \langle \nu_i(\vec{k}, \sigma) | \nu_a \rangle\ . \tag{I-2.7}$$

Here, we use the eigenvalues of $\hat{H}$ and $|\hat{\vec{P}}|$, energy E and momentum $|\vec{k}|$, of the eigenstates ν_i. In the massless limit, neutrinos propagate with exactly the speed of light, such that $|\vec{x}| = t$, meaning that E and $|\vec{k}|$ cancel exactly. Therefore, in the massless limit, the matrix element vanishes and there is no flavor oscillation. For light, ultra-relativistic neutrinos, however, we can expand

$$E_i = \sqrt{|\vec{k}|^2 + m_i^2} = |\vec{k}| \left(1 + \frac{m_i^2}{2|\vec{k}|} + \cdots\right) \approx |\vec{k}| + \frac{m_i^2}{2E} + \cdots . \tag{I-2.8}$$

Thus, the $|\vec{k}|$ cancels between the Hamiltonian and the translation operator, and after some simplifications, we get

$$\langle \nu_b | \nu_a(\vec{x}, t) \rangle = e^{i\,\xi} \sum_i e^{-i\, m_i^2\, L/(2E)} \underbrace{\langle \nu_b | \nu_i \rangle}_{\equiv U_{bi}} \langle \nu_i | \nu_a \rangle \tag{I-2.9}$$

$$= e^{i\,\xi} \sum_i e^{-i\, m_i^2\, L/(2E)}\, U_{bi}\, U_{ai}^* \,, \tag{I-2.10}$$

where ξ is some phase, $L = |\vec{x}|$ is the distance traveled by the neutrino, and we have defined the leptonic mixing matrix U. This mixing matrix is unitary, as probabilities need to be conserved,

$$\delta_{ij} \stackrel{!}{=} \langle \nu_i | \nu_j \rangle = \langle \nu_i | \underbrace{\sum_a |\nu_a\rangle \langle \nu_a|}_{= \mathbb{1}} |\nu_j \rangle = \sum_a U_{ai}^*\, U_{aj} \,. \tag{I-2.11}$$

From this, we obtain the unitarity condition on the leptonic mixing matrix,

$$U^\dagger\, U = \mathbb{1} \tag{I-2.12}$$

In the next step, we obtain the probability of the neutrino of energy E being in the flavor state b after having traveled a distance L. To get this oscillation probability, we take the absolute square of eq. (I-2.10), and obtain

$$P_{\nu_a \longrightarrow \nu_b}(E, L) = |\langle \nu_b | \nu_a(\vec{x}, t) \rangle|^2 \tag{I-2.13}$$

$$= \sum_{i,\,j} e^{-i\,(m_i^2 - m_j^2)\, L/(2E)}\, U_{bi}\; U_{bj}^*\, U_{aj}\, U_{ai}^* \,. \tag{I-2.14}$$

Let us inspect this equation; we see that it depends on the mixing matrix elements between the mass eigenstates and the flavor eigenstates, the energy of the neutrino, the distance traveled, and the mass square differences of the mass eigenstates. This tells us several things,

1. if mass and flavor eigenbasis are the same, the mixing matrix elements are Kronecker deltas, and the probability vanishes;

2. the oscillation frequency depends on the oscillation length $\sim 2E/\Delta m^2$, meaning that at least some neutrinos have to be massive, and not mass-degenerate;

3. if $L \longrightarrow 0$, $P_{\nu_a \longrightarrow \nu_b}(E,\, 0) = \delta_{ab}$ due to the unitarity of U; and

4. as L increases, the probability oscillates, hence giving this phenomenon the name of neutrino oscillations.

To understand eq. (I-2.14) better, let us consider the simplified case of just two flavors. For many experiments, this is a valid approximation—recall that for solar neutrinos, mostly electron and muon neutrinos played a role, and for atmospheric neutrinos only muon and tau neutrinos did. In this case, the mixing matrix is given by the simplified 2×2 rotation matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\,, \tag{I-2.15}$$

and we obtain for the probability of finding a different flavor than we started with

$$P_{\nu_a \longrightarrow \nu_b \neq \nu_a}(E,\, L) = \sin^2(2\theta)\, \sin^2\Big(\frac{\Delta m^2\, L}{4E}\Big)\,. \tag{I-2.16}$$

Here, once again, we see that

1. if there is no mixing between the flavor states, the oscillation probability vanishes;

2. if the neutrinos are massless, or mass-degenerate, the oscillation probability also vanishes;

3. in the limit $L \longrightarrow 0$, once again, the probability vanishes; and

4. the probability oscillates as L increases.

To get a feeling for the relevant scales involved, we can express the pertinent fraction of length, energy, and mass squared in terms of their units,

$$\frac{\Delta m^2\; L}{4E} \,\sim\, 1.27\;\frac{\Delta m^2\big[\text{eV}^2\big]\; L\big[\text{m}\big]}{E\big[\text{Mev}\big]}\,, \tag{I-2.17}$$

which already encapsulates the fact that in general, the mass squared differences of the neutrinos are much smaller than their energies.

Note that if we are interested in measuring the oscillations of anti-neutrinos, we need to replace $U \longrightarrow U^*$ in eq. (I-2.14). This offers us an opportunity to verify whether neutrinos and anti-neutrinos behave differently, i.e., such discrepancies offer signs of CP violation in the neutrino sector.

Furthermore, the discussion here applied to neutrino oscillations in the vacuum. When they propagate through matter, additional effects occur due to their interactions with the electrons present in matter; this is known as the MSW effect (named after S. Mikheyev, A. Smirnov, and L. Wolfenstein). In practice, this effect changes the in-medium mixing angles and mass eigenvalues—and thus mass splittings—and is important when evaluating oscillation experiments with in-medium propagation of the neutrinos. For instance, this effect occurs for solar neutrinos as they travel from the sun's core to its surface.

Before moving on to present the current experimental limits, let us show the form of the leptonic mixing matrix, U. In can be parametrized analogously to the CKM matrix of the quark sector by multiplying three rotation matrices, one of which contains a phase. This encapsulates the fact that we can absorb five of the six phases contained in a general 3×3 unitary matrix into the lepton fields, such that we are left with three angles and one phase. In the Majorana case, however, we can only remove two phases, due to the Majorana condition prohibiting such field redefinitions. Overall, defining $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$, we can write the leptonic mixing matrix as

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\,\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\,\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{I-2.18}$$

$$= \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\,\delta} \\ -c_{23}\, s_{12} - s_{23}\, c_{12}\, s_{13}\, e^{i\,\delta} & c_{23}\, c_{12} - s_{23}\, s_{12}\, s_{13}\, e^{i\,\delta} & s_{23}\, c_{13} \\ s_{23}\, s_{12} - c_{23}\, c_{12}\, s_{13}\, e^{i\,\delta} & -s_{23}\, c_{12} - c_{23}\, s_{12}\, s_{13}\, e^{i\,\delta} & c_{23}\, c_{13} \end{pmatrix} . \tag{I-2.19}$$

In the Majorana case, we define $V = U\,K$, where K contains the additional Majorana phases,

$$K = \begin{pmatrix} e^{i\,\alpha_1/2} & 0 & 0 \\ 0 & e^{i\,\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{I-2.20}$$

Note that in the oscillation probability of eq. (I-2.14), the Majorana phases drop out since we take the absolute value squared of the matrix element.

Furthermore, note that the leptonic mixing matrix is also often referred to as MNS (Maki, Nakagawa, Sakata) or PMNS (Pontecorvo, Maki, Nakagawa, Sakata) matrix.

I-2.1.3 Experimental Limits on Neutrino Parameters, and Mass Hierarchy

Having discussed neutrino oscillations, we now present the current experimental limits on neutrino parameters, taken from [3, 4], in tab. (I-2.1). The results are given both with and without atmospheric data from Super-Kamiokande, as well as for both the Normal Ordering (NO), and Inverse Ordering (IO) hierarchies. We will discuss these hierarchies shortly.

Note that, following our previous discussions on the solar neutrino problem, and atmospheric neutrinos, different types of experiments are sensitive to different mixing angles and mass splittings. We can understand this considering that depending on the creation mechanism and detector location,

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.7$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
without SK atmospheric data	$\sin^2\theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$
	$\theta_{12}/^\circ$	$33.44^{+0.78}_{-0.75}$	$31.27 \to 35.86$	$33.45^{+0.78}_{-0.75}$	$31.27 \to 35.87$
	$\sin^2\theta_{23}$	$0.570^{+0.018}_{-0.024}$	$0.407 \to 0.618$	$0.575^{+0.017}_{-0.021}$	$0.411 \to 0.621$
	$\theta_{23}/^\circ$	$49.0^{+1.1}_{-1.4}$	$39.6 \to 51.8$	$49.3^{+1.0}_{-1.2}$	$39.9 \to 52.0$
	$\sin^2\theta_{13}$	$0.02221^{+0.00068}_{-0.00062}$	$0.02034 \to 0.02430$	$0.02240^{+0.00062}_{-0.00062}$	$0.02053 \to 0.02436$
	$\theta_{13}/^\circ$	$8.57^{+0.13}_{-0.12}$	$8.20 \to 8.97$	$8.61^{+0.12}_{-0.12}$	$8.24 \to 8.98$
	$\delta_{CP}/^\circ$	195^{+51}_{-25}	$107 \to 403$	286^{+27}_{-32}	$192 \to 360$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.514^{+0.028}_{-0.027}$	$+2.431 \to +2.598$	$-2.497^{+0.028}_{-0.028}$	$-2.583 \to -2.412$
		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 7.1$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$
	$\theta_{12}/^\circ$	$33.44^{+0.77}_{-0.74}$	$31.27 \to 35.86$	$33.45^{+0.78}_{-0.75}$	$31.27 \to 35.87$
	$\sin^2\theta_{23}$	$0.573^{+0.016}_{-0.020}$	$0.415 \to 0.616$	$0.575^{+0.016}_{-0.019}$	$0.419 \to 0.617$
	$\theta_{23}/^\circ$	$49.2^{+0.9}_{-1.2}$	$40.1 \to 51.7$	$49.3^{+0.9}_{-1.1}$	$40.3 \to 51.8$
	$\sin^2\theta_{13}$	$0.02219^{+0.00062}_{-0.00063}$	$0.02032 \to 0.02410$	$0.02238^{+0.00063}_{-0.00062}$	$0.02052 \to 0.02428$
	$\theta_{13}/^\circ$	$8.57^{+0.12}_{-0.12}$	$8.20 \to 8.93$	$8.60^{+0.12}_{-0.12}$	$8.24 \to 8.96$
	$\delta_{CP}/^\circ$	197^{+27}_{-24}	$120 \to 369$	282^{+26}_{-30}	$193 \to 352$
	$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
	$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.517^{+0.026}_{-0.028}$	$+2.435 \to +2.598$	$-2.498^{+0.028}_{-0.028}$	$-2.581 \to -2.414$

Table I-2.1.: Experimental limits on neutrino parameters in normal ordering and inverted ordering, as well as with and without atmospheric data from Super-Kamiokande; taken from [3, 4]

- we start with different neutrino flavors;
- the neutrinos have different energy ranges; and
- the distance between creation and detection site is different.

In particular, we find that

- solar experiments are particularly sensitive to θ_{12} and Δm^2_{12};
- atmospheric experiments are particularly sensitive to θ_{23} and Δm^2_{32}; and
- reactor experiments are particularly sensitive to θ_{13}.

We see from tab. (I-2.1) that the solar and atmospheric mixing angles—approximately corresponding to θ_{12} and θ_{23}, respectively—are very large, especially compared to the quark sector. The third mixing angle, θ_{13}, on the other hand, is comparatively small. Since the CP phase δ_{CP} always appears together with $\sin\theta_{13} \sim \theta_{13}$, a small θ_{13} means that it is difficult to measure δ_{CP} accurately, as the small angle suppresses its effects.

As for the mass splittings, we see that since there are two nonzero mass splittings, at least two of the neutrinos have to be massive. Furthermore, we observe that there are two neutrino masses that are relatively close to each other, with $|\Delta m^2_{21}| = |m^2_{\nu,2} - m^2_{\nu,1}| \sim 7.4 \times 10^{-5}\,\mathrm{eV}^2$, while the splitting to the third is larger by two orders of magnitude, $|\Delta m_{3\ell}|^2 = |m^2_{\nu,3} - m^2_{\nu,1\,\mathrm{or}\,2}| \sim 2.5 \times 10^{-5}\,\mathrm{eV}^2$. Since the splitting between two of the masses is so much smaller than the splitting to the third, the large splitting is only denoted as mass square difference between the third mass with either of the other two. Another important aspect is that of mass hierarchies, which we did not discuss until now.

In addition to previously discussed points, the mass hierarchy of neutrinos is an important characteristic for experimental limits and our understanding of the neutrino sector in general. In particular, from the oscillation probability in the two flavor case, eq. (I-2.16), we see that we can only measure the absolute value of the mass square differences in oscillation experiments. Since we know from the data that

$$|\Delta m_{31}|^2 \sim |\Delta m_{32}|^2 \gg |\Delta m_{21}|^2\,, \qquad \text{(I-2.21)}$$

we are left with two possibilities to order the masses:

$$m_3 > m_2 > m_1\,, \qquad \text{Normal Ordering,} \quad \text{or} \qquad \text{(I-2.22)}$$

$$m_2 > m_1 > m_3\,, \qquad \text{Inverted Ordering.} \qquad \text{(I-2.23)}$$

Note that in general, we define $m_2 > m_1$. We show a pictorial representation of the mass hierarchies in fig. (I-2.4).

There are two more relevant limits on neutrino masses we would like to mention; direct mass measurement in beta decay, and constraints on the sum of neutrino masses from cosmology. Furthermore, neutrinoless double beta decay is also an important measure of Majorana masses of neutrinos, but we will not cover this here. The first additional constraint we would like to discuss, comes from the KATRIN experiment, which determines an effective neutrino mass

$$m^2_\nu = \sum_i |U_{ei}|^2\, m^2_i\,, \qquad \text{(I-2.24)}$$

in the beta decay of tritium—a hydrogen isotope with two neutrons. By measuring the kinetic energy distribution of the resulting electron, it is possible to extract the effective neutrino mass. In particular, at high kinetic energies, the spectrum exhibits a dip due to the missing energy required to account for the neutrino mass. In the presence of neutrino mixing, only the effective mass of eq. (I-2.24) can be determined, as the electron neutrino from the beta

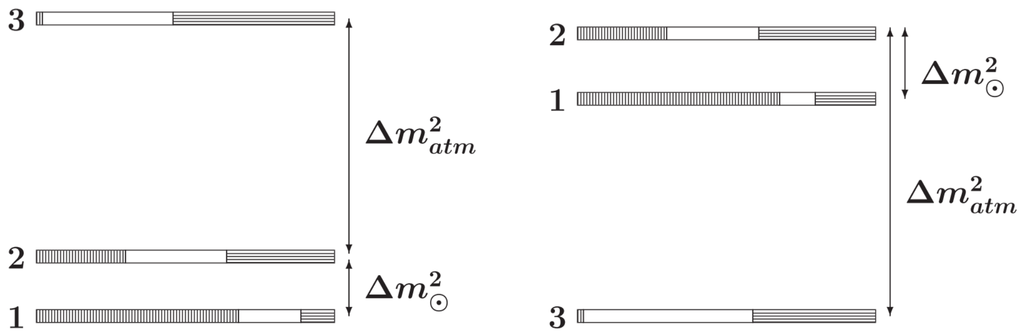


Figure I-2.4.: Neutrino mass hierarchies and approximate mixing to the flavor states; the left shows Normal Ordering, the right Inverted Ordering; the vertically dashed regions indicate the amount of mixing to the electron flavor, the open regions, to the muon, and the horizontally dashed, to the tau; $\Delta m^2_{atm} = \Delta m^2_{32}$ is the atmospheric mass splitting, and $\Delta m^2_{\odot} = \Delta m^2_{21}$, the solar mass splitting; taken from [25]

decay is a superposition of the mass eigenstates. The KATRIN collaboration sets a limit on the effective mass of

$$m_\nu < 0.8\,\mathrm{eV}\,. \tag{I-2.25}$$

This result was published in [5].

Another important limit comes from cosmological measurements. This discussion is based on [27]; see also, e.g., [28]. On one hand, neutrinos act for the most part as radiation because they are ultra-relativistic for most of the universe's evolution. However, nonzero neutrino masses lead to them becoming nonrelativistic once the temperature of the universe drops below their mass scale. However, their energetic distribution function is given by a rescaled version of the ultra-relativistic case. From this, their total contribution to the energy density of the universe can be inferred. Their relative energy density is given by

$$\Omega_\nu = \frac{\sum m_\mu \, n_\nu(t_0}{\rho_c} \approx \frac{\sum m_\nu}{94\,h^2\,\mathrm{eV}}\,, \tag{I-2.26}$$

where $n_\nu(t_0)$ is the distribution function of neutrinos today, ρ_c is the critical density—the total energy density of the universe today—and h is the reduced Hubble constant that parametrizes the expansion rate today. This energy density then has various effects on cosmology, which can be measured to derive an upper limit on the sum of neutrino masses. For instance, massive neutrinos can act as an admixture of hot dark matter, which affects structure formation; in particular, massive neutrinos suppress structure formation at small scales. Furthermore, they can affect the anisotropies of the Cosmic Microwave Background, as this depends on the total energy density of relativistic degrees of freedom in the early universe. The eBOSS collaboration provides a cosmological limit on the sum of neutrino masses by combining measurements from various experiments and effects [6]. For the standard cosmological model (ΛCDM) with massive neutrinos, their resulting upper limit is given by

$$\sum m_\nu < 0.111\,\text{eV}\,. \tag{I-2.27}$$

Thus, we conclude this section on the motivation and evidence for neutrino masses. Next, will consider how to implement neutrino masses in particular models.

I-2.2 Neutrino Mass Models in Standard Model Extensions

After covering the evidence suggesting the existence of nonzero neutrino masses, as well as the current experimental limits, let us now turn our attention to the description and implementation of massive neutrinos in extensions of the Standard Model.

I-2.2.1 Type-I Seesaw Models of Neutrino Mass Generation

Arguably the most well-known framework of generating neutrino masses is via so-called seesaw models. To begin, there are three main types of seesaw mechanisms. While there are others as well, we will not cover them here.

The simplest type of seesaw is the Type-I seesaw model. It is also the mechanism we mainly focus on in this work for specific models and relevant scales. In Type-I seesaw models, we add heavy, right-handed neutrinos N_i to the SM. These new fields are SM singlets, and thus may have a Majorana mass term. We then couple the right-handed neutrinos to the left-handed ones via a Yukawa coupling with the Higgs-doublet, which will lead to nonzero masses for the SM neutrinos after Electroweak Symmetry Breaking (EWSB). The relevant Lagrangian part for this seesaw is given by

$$-\mathcal{L}_{seesaw} = Y_{\nu,\,ij}\,\overline{l_i}\,\varepsilon\,\phi^*\,N_j + \frac{1}{2}\,M_{ij}\,\overline{N_i^C}\,N_j \;+\text{h.c.}\,, \tag{I-2.28}$$

where Y_ν is the Yukawa coupling for the neutrinos, l_i are the lepton-doublets, ε is the totally antisymmetric 2×2 tensor in $SU(2)_L$ space, N_i are the right-handed neutrinos, and M is the Majorana mass matrix of N_i.

After EWSB, the coupling we can replace the Higgs-doublet by the Higgs vacuum expectation value (vev) and thus obtain a Dirac mass matrix, m_D. To see how the total mass matrix arises in this case, let us focus on just one flavor of neutrinos. Thus, we can rewrite the Lagrangian as

$$-\mathcal{L}_{seesaw} \overset{\text{EWSB}}{=} \frac{1}{2}\, m_D \left(\overline{\nu}\, N + \overline{N^C}\, \nu^C\right) + \frac{1}{2}\, M\, \overline{N^C}\, N \;\; + \text{h.c.} \tag{I-2.29}$$

$$= \frac{1}{2} \begin{pmatrix} \overline{\nu} & \overline{N^C} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \begin{pmatrix} \nu^C \\ N \end{pmatrix} \; + \text{h.c.}\,, \tag{I-2.30}$$

We then need to diagonalize this matrix, as ν and N are not the mass eigenstates—we see this from the fact that the mass matrix in eq. (I-2.30) is not diagonal. To diagonalize the matrix, we calculate

$$0 = \det \begin{pmatrix} -m_\pm & m_D \\ m_D & M - m_\pm \end{pmatrix} \tag{I-2.31}$$

$$= -m_\pm \left(M - m_\pm\right) - m_D\, m_D \tag{I-2.32}$$

$$= -m_\pm^2 - M\, m_\pm - m_D\, m_D\,, \tag{I-2.33}$$

from which we obtain

$$m_\pm = \frac{M \pm \sqrt{M^2 + 4 m_D\, m_D}}{2}\,. \tag{I-2.34}$$

We now expand this for large $M \gg m_D$, which yields

$$m_\pm = \frac{M \pm M \left(1 + \frac{1}{2}\, 4 m_D\, m_D\, M^{-1} + \cdots\right)}{2} \approx \begin{cases} M\,, & \text{for } \oplus \\ -\frac{m_D^2}{M}\,, & \text{for } \ominus \end{cases} \tag{I-2.35}$$

We obtain the corresponding mass eigenstates by solving the eigenvalue equation

$$\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = m_\pm \begin{pmatrix} p \\ q \end{pmatrix}\,. \tag{I-2.36}$$

We can consider the top row for $m_\pm$, setting either q or p to 1, and the solving the equations

$$m_D\, q \overset{m_+}{=} M\, p \xrightarrow{q=1} p = \frac{m_d}{M} \tag{I-2.37}$$

$$m_D\, q \overset{m_-}{=} -\frac{m_D^2}{M}\, p \xrightarrow{p=1} q = -m_D M\,. \tag{I-2.38}$$

Thus, the resulting mass eigenstates are given by

$$\tilde{N} \approx N + \frac{m_D}{M}\, \nu^C \quad \Longrightarrow \tilde{N} \approx N \tag{I-2.39}$$

$$\tilde{\nu} \approx \nu - \frac{m_D}{M}\, N^C \quad \Longrightarrow \tilde{\nu} \approx \nu\,. \tag{I-2.40}$$

From the above results, we also observe the reason behind the naming "seesaw" mechanism. The heaviness of the right-handed neutrino is responsible for the small masses of the left-handed neutrinos. We can also take this opportunity to make some rough dimensional estimations for the relevant scales. For m_D, we can, e.g., take $m_D \sim 100\,\mathrm{GeV}$, which is motivated by the electroweak scale, given that m_D is determined from the Higgs vev. Considering our previous experimental results, we then take a well-motivated value of $m_D^2/M \sim 0.1\,\mathrm{eV}$ for the left-handed neutrino mass. From these two, we can calculate the necessary mass scale for the right-handed neutrinos, and find

$$\left.\begin{aligned} m_D &\sim 100\,\mathrm{GeV} \\ \frac{m_D^2}{M} &\sim 0.1\,\mathrm{eV} \end{aligned}\right\} \quad \Longrightarrow \quad M \sim 10^{14}\,\mathrm{GeV} \quad ! \tag{I-2.41}$$

This means that to explain the small masses of the SM neutrinos via a Type-I seesaw mechanism, we would need right-handed neutrinos with Majorana masses of the order of 10^{14} GeV! Therefore, we see that such right-handed neutrinos are extremely heavy, which might explain why we have not detected them yet. Furthermore, this also emphasizes that the mass eigenstates and chirality eigenstates—the left- and right-handed neutrinos—are only slightly different, and can for the most part be taken to be the same.

Note that such heavy Majorana neutrinos as the N can also be used to explain the matter-antimatter asymmetry of the universe via a process called leptogenesis, which also makes them interesting beyond neutrino mass generation.

Going back to the case with multiple neutrino flavors, our parametrization of the Type-I seesaw mechanism yields a Majorana mass matrix for the left-handed neutrinos given by

$$m_\nu = -m_D^*\, M^{-1}\, m_D^\dagger = -\frac{v_\phi^2}{2}\, Y_\nu^*\, M^{-1}\, Y_\nu^\dagger\,. \tag{I-2.42}$$

In the last step, we have expressed m_D in terms of the Yukawa coupling matrices and the vev of the Higgs field. Note also that the negative sign is not a problem for causality, as it corresponds to a phase we.

Before moving on to the other two seesaw types, let us briefly consider the interaction diagram that gives rise to the seesaw formula of eq. (I-2.42). Since it gives us a Majorana mass term, we know that it needs to be a lepton-number-violating interaction. From the form of the mass matrix, we can deduce that we need two insertions of the Yukawa coupling. Lastly, the inverse Majorana mass matrix of the right-handed neutrinos signals the presence of a propagator, as fermion propagators are of the form

$$\frac{i}{\not{p} - M}\,, \tag{I-2.43}$$

where $\not{p}$ is the fermion's momentum contracted with gamma matrices, and M the fermion's mass. Indeed, the pertinent diagram giving rise to Majorana masses for the left-handed neutrinos via the Type-I seesaw mechanism is given precisely by a combination of the components above.

We show the Feynman diagram of this interaction in fig. (I-2.5)—we do not show the diagram with crossed ϕ legs, as we are only interested in the general structure here. Note that we have shown the interaction before EWSB, where it involves the lepton-doublets as a whole. Furthermore, the transposition $Y_\nu^\dagger \longrightarrow Y_\nu^*$ originates in the computation of the diagram due to its fermion-number-violating nature. We will touch on this soon.

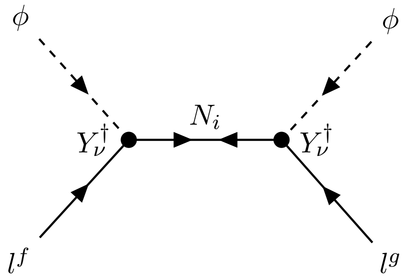


Figure I-2.5.: Feynman diagram for a Type-I Seesaw Mechanism; l^f and l^g are lepton-doublets of flavors f and g, respectively; N_i are heavy, right-handed neutrinos, ϕ is the Higgs-doublet, and Y_ν the neutrino Yukawa coupling

I-2.2.2 Type-II and Type-III Seesaw Models of Neutrino Mass Generation

Let us now briefly discuss the other two main types of seesaw mechanisms.

In Type-II seesaw models, we add a heavy, scalar $SU(2)_L$ triplet Δ to our theory, whose most relevant interaction terms are given by

$$-\mathcal{L} \supset \frac{1}{2}\, Y^{\Delta}_{\nu,\,gf}\, \overline{l^C_g}\, \left(\Delta^a\, \tau^a\right) \varepsilon\, l^f + \frac{1}{2}\, c\, \phi^\dagger \left(\left(\Delta^a\right)^* \tau^a\right) \varepsilon\, \phi^* \ + \text{h.c.}\,, \tag{I-2.44}$$

where we defined the Yukawa coupling matrix Y_ν^Δ, and the cubic coupling constant of Δ to the Higgs-doublet, c. Furthermore, τ^a are the Pauli matrices. In as similar way as for the Type-I seesaw mechanism, the Majorana mass matrix arises from a combination of these couplings, the propagator of the new field, and—after EWSB—the Higgs vev. The corresponding Majorana mass matrix is more straightforward to obtain than in the Type-I seesaw, since we do not have right-handed neutrinos to mix with. Note that additional complications arise in the scalar sector, however. The resulting mass matrix after EWSB for the left-handed neutrinos is given by

$$m_\nu = \frac{v_\phi^2\, c\, Y_\nu^\Delta}{2 M_\Delta^2}\,, \tag{I-2.45}$$

where M_Δ is the mass of the triplet scalar. We show the corresponding Feynman diagram in fig. (I-2.6)

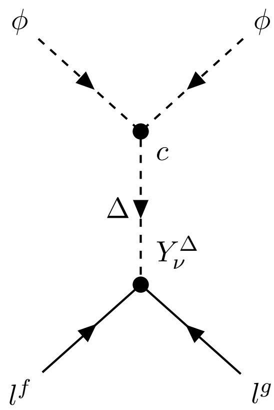


Figure I-2.6.: Feynman diagram for a Type-II Seesaw Mechanism; l^f and l^g are lepton-doublets of flavors f and g, respectively; Δ is a heavy, $SU(2)_L$ scalar, ϕ is the Higgs-doublet, Y_ν^Δ the Yukawa coupling of the lepton-doublets to Δ, and c the cubic coupling of the Higgs-doublets to Δ

Note that these types of seesaw models may also be extended with right-handed neutrinos for Type-I+II hybrid models.

Lastly, in Type-III seesaw models, instead of an $SU(2)_L$ triplet scalar, we add a triplet lepton $\Sigma = \Sigma^a\, \tau^a/2$ [29]. The pertinent Lagrangian terms are given by

$$-\mathcal{L} \supset \left(Y^{\Sigma}_{\nu,\,i}\, \overline{l}_i\, \Sigma^C \;\; \phi^* \;\; + \text{h.c.}\right) + M_\Sigma \; \text{Tr}\left(\Sigma^\dagger\, \Sigma\right), \tag{I-2.46}$$

where we introduced the Yukawa coupling Y_ν^Σ for the new Σ lepton with the lepton-, and Higgs-doublet, and the term with the trace corresponds to a mass term for Σ. Analogously to the previous two models, we obtain a mass matrix for the neutrinos given by

$$m_\nu = -\frac{v_\phi^2}{2}\, Y_\nu^*\, M_\Sigma^{-1}\, Y_\nu^\dagger\,, \tag{I-2.47}$$

We show the corresponding Feynman diagram of in fig. (I-2.7)—note again that we do not add the diagram with crossed Higgs-doublet legs, as we are only interested in the general structure here.

There are many further models to generate neutrino masses at tree level as we have seen it here, but also radiatively via, e.g., one-loop interactions. Furthermore, there are also models for neutrino masses using Grand Unified Theories, e.g., $SO(10)$. However, we will not discuss these here.

By inspecting the three seesaw models we have discussed, we notice one common trait. Namely, they all induce mass terms for the neutrinos by coupling two external lepton-doublets with two external Higgs-doublets. After EWSB, this then induces the neutrino masses. In fact, for all of these models, the neutrino masses arise in the limit of large masses of the intermediate fields; this is not a coincidence. There is an underlying reason for why this occurs, and why models

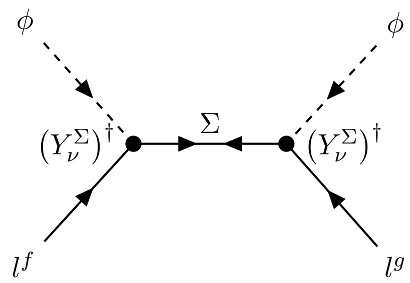


Figure I-2.7.: Feynman diagram for a Type-I Seesaw Mechanism; l^f and l^g are lepton-doublets of flavors f and g, respectively; Σ is a heavy, $SU(2)_L$ doublet lepton, ϕ the Higgs-doublet, and Y_ν^Σ the Yukawa coupling of Σ with the lepton-, and Higgs-doublet

frequently rely on this paradigm. This leads us to the effective description of neutrino masses and the dimension five Weinberg-Operator.

I-2.3 Effective Description of Neutrino Masses

We have seen before how we can induce neutrino masses via interaction with new fields, and the Higgs-doublet vev after EWSB. In particular, we observed that the mass terms arise from a common type of diagram that involves two incoming lepton-, and Higgs-doublets. This is because we assumed the new fields to be heavy, which led us to an effective, low-energy description of neutrino masses in the limit of $M \gg p$, where M is the new field's mass, and p its momentum. Let us now delve into this further.

I-2.3.1 The Weinberg-Operator

I-2.3.1.1 Effective Field Theories and Effective Operators

First, let us briefly review effective operators—see, for instance, [30, 31, 32, 33]. Effective operators are nonrenormalizable, which means that to renormalize divergent loop diagrams in which they appear, requires consecutively adding further counterterms to cancel the divergence—note that we will discuss renormalization in more detail in the next chapter. For instance, let us imagine we had an effective interaction term

$$-\mathcal{L}_{\text{eff}} = \lambda_6\, \phi^6\,, \tag{I-2.48}$$

where ϕ is some real scalar field. In this case, a one-loop diagram with two insertions of this coupling, as shown in fig. (I-2.8a), would require a counterterm of the form

$$-\mathcal{C} = c_8\, \phi^8\,. \tag{I-2.49}$$

The diagram to this counterterm is shown in fig. (I-2.8b). Note that we can effectively obtain this counterterm by pinching the intermediate propagators. However, to cancel the divergences from the one-loop diagrams of the emerging ϕ^8 coupling, we would have to introduce a ϕ^{12} counterterm, etc. Thus, consecutively, we would have to add more and more counterterms to the Lagrangian, making this theory nonrenormalizable—i.e., we cannot renormalize it with a finite number of counterterms.

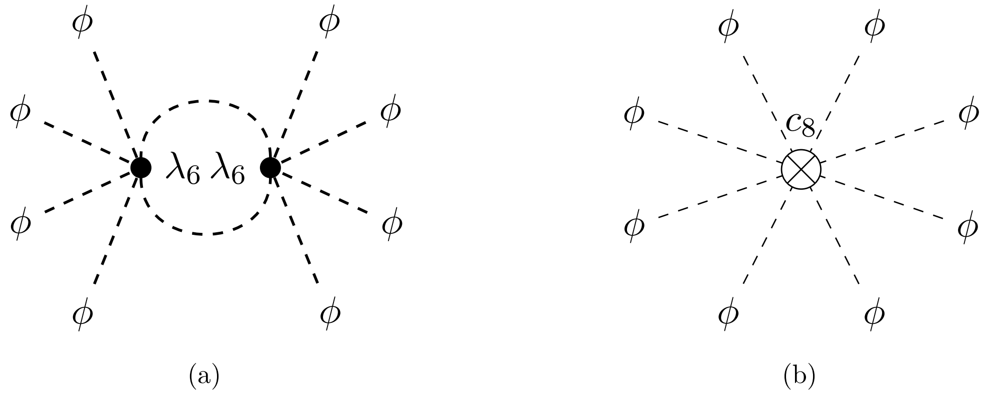


Figure I-2.8.: One-loop diagram for two insertions of an effective ϕ^6 coupling λ_6 in fig. (I-2.8a), and counterterm c_8 diagram for two insertions of an effective ϕ^6 coupling λ_6 in fig. (I-2.8b)

In practice, we can identify effective interactions by a coupling constant with *negative mass dimension*,

$$\mathcal{L} \supset \frac{c}{\Lambda^n}\,\mathcal{O}(\phi_i)\,, \tag{I-2.50}$$

where c is a dimensionless constant, Λ is a dimensionful scale raised to some power n, and $\mathcal{O}(\phi_i)$ is an operator of some fields ϕ_i. This also tells us that we may renormalize non-renormalizable, effective theories as long as we remain below the scale Λ. Generally, this scale is the scale up to which the effective operator provides a valid, predictive description of physics. Therefore, we perform an expansion in $1/\Lambda$, and take Λ to a large scale. Thus, if we are only interested in processes up to a given precision—parametrized by a certain power in Λ—then we can use the effective interaction terms up to that order, and even renormalize them. For instance, the λ_6 coupling has a negative mass dimension of -2, so if we are only interested in process up to order $1/\Lambda^2$, then we neglect diagrams such as in fig. (I-2.8a), and thus do not need the counterterm of fig. (I-2.8b). Hence, our theory is perfectly predictive as long as we remain within its area of validity.

While they can be used in perturbative calculations similarly to renormalizable operators, nonrenormalizable operators parametrize effective interactions that generally stem from low-energy descriptions of high-energy theories. The high-energy region is called the ultraviolet (UV) region, and is given by the scale Λ. The fact that effective operators only offer a reasonable description of phenomena up to this scale stems from the fact that the effective operators are obtained from "integrating out" heavy fields below the UV scale characteristic for their masses. The idea is that they essentially become inactive, or rather, they do not propagate anymore. The term"integrate out" comes from the path integral formulation of Quantum Field Theory (QFT), wherein these fields are integrated over, and thus drop from the explicit description of the model. In diagrams where the heavy fields propagate, their propagators are pinched, or in other words, contracted to a point—the approximation is thus in $p \ll M$, i.e., the momentum being much smaller than the particle's mass. This pinched propagator then leads to an effective contact interaction with a coupling $\sim 1/M^2$, hence the inverse mass dimension of effective operators' couplings.

With the knowledge of the points discussed just now, let us return to neutrino masses, and apply it to their effective description.

I-2.3.1.2 The Effective Neutrino Mass Operator

As teased previously, one common train of the seesaw models was that all the diagrams that eventually lead to Majorana mass matrices for the neutrinos involved two external lepton-, and Higgs-doublets. Furthermore, we saw that the resulting mass matrices were inverse in the intermediate particle's mass, which came from its propagator in the respective diagrams. The underlying paradigm to both of these is the effective, low-energy description of neutrino masses below the mass scale of the new, heavy fields. In particular, in the Effective Field Theory (EFT) of the Standard Model, which parametrizes effective Operators that are compatible with the SM symmetries contains precisely the effective operator that emerges from integrating out the heavy fields introduced in the seesaw models. This operator is known as the Weinberg-Operator, and is given by

$$\mathcal{L}_{\kappa} = \frac{1}{4}\, \kappa_{gf}\, \overline{l_c^{g,C}}\, \varepsilon^{cd}\, \phi_d\, l_b^f\, \varepsilon^{ba}\, \phi_a \;+ \text{h.c.}\,, \tag{I-2.51}$$

where κ_{gf} is the coupling matrix in flavor-space, the superscripts are flavor indices, and the subscripts $SU(2)_L$ indices. We have taken this specific form from [34]. Note that we will often use κ representatively for the Weinberg-Operator, as it encapsulates most aspects we are interested in here. Furthermore, we will frequently drop the $SU(2)_L$ indices for clarity when its structure is not the focus of the discussion. We show the Feynman diagram of the effective interaction of eq. (I-2.51) in fig. (I-2.9:).

From the flavor structure of eq. (I-2.51),

$$\mathcal{L}_{\kappa} \sim l^g\, l^f\, \phi\, \phi\,, \tag{I-2.52}$$

we see that κ is a symmetric matrix in flavor-space, since antisymmetric components vanish,

$$\kappa_{gf}\, l^g\, l^f \overset{!}{=} -\kappa_{fg}\, l^g\, l^f = -\kappa_{gf}\, l^f\, l^g = -\kappa_{gf}\, l^g\, l^f \tag{I-2.53}$$

$$= 0\,. \tag{I-2.54}$$

Thus, we find that

$$\kappa^T \,=\, \kappa\,. \tag{I-2.55}$$

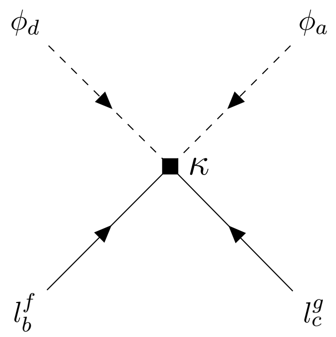


Figure I-2.9.: Feynman diagram of the Weinberg-Operator

Since we are interested in neutrino masses, let us see how they arise from eq. (I-2.51). The operator is invariant under the SM gauge group, and thus also under $SU(2)_L$, which we see from the fact that it is written in terms of the lepton-, and Higgs-doublet. However, let us pick out of all interaction terms contained in this operator the one that contains two neutrinos,

$$\mathcal{L}_{\kappa,\nu} = \frac{1}{4}\,\kappa_{gf}\,\overline{\nu^C}^g\,\nu^f\,\phi^0\,\phi^0\; + \text{h.c.}\,. \tag{I-2.56}$$

We see that it is also quadratic in the neutral Higgs field! This means that after EWSB, we can take $\phi^0 \longrightarrow v_\phi/\sqrt{2}$, and obtain a Majorana mass term for the neutrinos,

$$\mathcal{L}_{\kappa,\nu} = \frac{1}{4}\,\kappa_{gf}\,\overline{\nu^C}^g\,\nu^f\,\phi^0\,\phi^0\; + \text{h.c.} \quad \xrightarrow{\text{EWSB}} \quad \frac{\kappa_{gf}\,v_\phi^2}{4}\,\frac{1}{2}\,\overline{\nu^C}^g\,v^f\; + \text{h.c.}\,. \tag{I-2.57}$$

We see that we do indeed get a Majorana mass matrix, given by $m_{\nu,\,gf} = \frac{1}{4}\,v_\phi^2\,\kappa_{gf}$. This also tells us that

The eigenvalues of κ correspond to the mass eigenvalues of the neutrinos.

Let us now turn to the connection between the effective Weinberg-Operator and the UV neutrino mass model we discussed in the previous section.

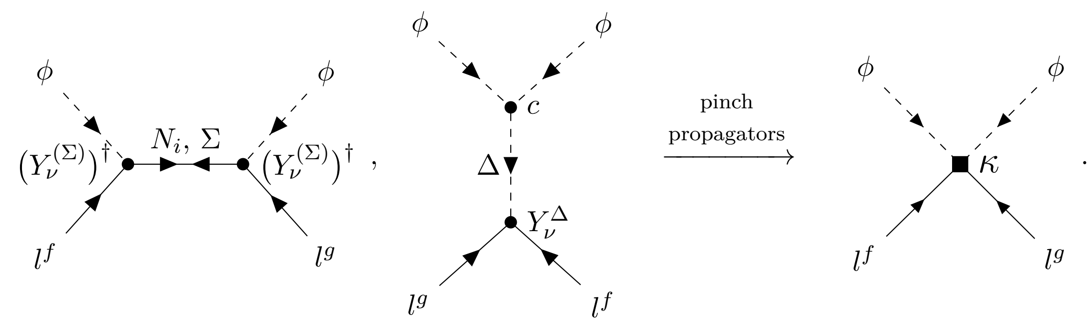
I-2.3.1.3 Relating the Effective and Fundamental Descriptions of Neutrino Masses

Inspecting the operator of eq. (I-2.51),

$$\mathcal{L}_{\kappa} \sim l^g\, l^f\, \phi\, \phi\,, \tag{I-2.58}$$

we can see that it does, in fact, exactly correspond to the external states in fig. (I-2.5), (I-2.6), and (I-2.7)! Namely, we have two incoming lepton-doublets—recall that $\overline{l^C} \sim \left(l^*\right)^\dagger = l^T$—and two Higgs-doublets. Note that we drop the $SU(2)_L$ indices here. Furthermore, if we pinch the propagators of the intermediate particles in any of the seesaw models we discussed, we obtain precisely the diagram of the Weinberg-Operator in fig. (I-2.9).

Generically, we see that taking integrating out the heavy fields of the seesaw mechanisms I-III yields



(I-2.59)

Therefore, any of the seesaw mechanisms provides a possible UV completion for the effective Weinberg-Operator, and vice versa, the Weinberg-Operator provides a good description of low-energy neutrino masses independently of what the UV theory is. Therein lies the power of effective operators. Another important aspect in particular for the Weinberg-Operator is that it is, in fact, the *only* SM gauge-invariant operator of dimension five—in terms of the UV scale this corresponds to $1/\Lambda$. Therefore, it is also the only effective operator to describe neutrino masses at the order $1/\Lambda$. Note that higher-dimensional operators may also describe and contribute to neutrino masses, but as they are of at least $\mathcal{O}(1/\Lambda^2)$, they are heavily suppressed compared to the Weinberg-Operator.

The fact that we can obtain the effective Weinberg-Operator from integrating out heavy fields also leads us to the concept of matching. Here, we calculate the externally identical diagram in both the effective theory, and the UV theory, and then require the two be equal in the limit $p \ll M$ for the intermediate particle—i.e., at the UV scale Λ. Let us, for instance, assume we are in a Type-I seesaw framework—we will follow [34] for this part. In this case, we equate the coupling matrix κ with the neutrino mass matrix of eq. (I-2.42) without the vev of the Higgs field. This is because we perform the matching before EWSB, and the Higgs-doublet appears explicitly in the definition of the Weinberg-Operator. We thus obtain

$$\kappa = -2\, Y_\nu^*\, M^{-1}\, Y_\nu^\dagger\,, \tag{I-2.60}$$

where the factor of 2 comes from the normalization of 1/4 in the definition of the Weinberg-Operator.

As mentioned before, the eigenvalues of κ correspond to the masses of the neutrino mass eigenstates; therefore, we would like to know how to diagonalize κ. Inspecting the form of eq. (I-2.51) in flavor-space, we have

$$\mathcal{L}_\kappa \sim \overline{l^C}\,\kappa\, l \sim l^T\,\kappa\, l\,. \tag{I-2.61}$$

Now, we perform the transformation that takes the lepton-doublet from the flavor basis to the mass basis,

$$l \longrightarrow U\, l\,, \tag{I-2.62}$$

and require that $\mathcal{L}_\kappa$ be invariant:

$$\mathcal{L}_\kappa \longrightarrow \tilde{\mathcal{L}}_\kappa \sim l^T\, U^T\, \tilde{\kappa}\, U\, l \overset{!}{=} l\,\kappa\, l\,, \tag{I-2.63}$$

where we have denoted the transformed quantities by a tilde. Thus, using the unitarity of U, and we find that κ transforms as

$$\kappa \;\longrightarrow\; U^*\,\kappa\, U^\dagger \equiv \kappa_{\text{diag}}\,. \tag{I-2.64}$$

Here, we have defined the diagonal coupling matrix κ_{diag}. The fact that κ transforms with U^* and $U^\dagger$ encapsulates its lepton-number-violating nature, as an overall phase cannot be freely absorbed via field redefinitions—lepton-number-conserving interactions transform with U and $U^\dagger$. Note that in this work, we follow the convention of transforming κ itself under flavor transformations. In some works, the U coming from transforming l is instead absorbed in κ to transform it, which leads to a definition of κ_{diag} as $U^T\,\kappa\, U$, instead. In terms of the neutrino masses, we thus have

$$\kappa_{\text{diag}} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} = \frac{4}{v_\phi^2} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}\,, \tag{I-2.65}$$

where we have defined the eigenvalues of κ as κ_i. Recall that the additional factor of 2 with respect to $2/v_\phi^2$ in the numerator comes from the normalization 1/4 in the definition of κ.

Before moving on to the next chapter, let us consider one further aspect important to this work and neutrino mass models, flavor symmetries, and in particular $U(1)$ flavor gauge extensions.

I-2.3.2 Flavor Symmetries and $U(1)$ Flavor Gauge Extensions of the Standard Model

Let us now consider the aspect of flavor symmetries. These are particularly interesting in the context of generating relations between entries of the mass matrix for neutrinos, which may explain, e.g., the large mixing angles observed in nature. One direction of these flavor symmetries is given by non-abelian, discrete groups—see, e.g., [8]. Another avenue is given by flavor gauge extensions, in particular $U(1)$ extensions are frequently considered. We will focus on these here.

Of the $U(1)$ flavor extensions, ones where the difference of two lepton flavors is gauged are often of particular interest. The reason for this lies in the fact that these can be gauged without introducing gauge anomalies, even if no new fermions are added [9]. The three candidates for this are

$$U(1)_{L_e - L_\mu}\,, \quad U(1)_{L_e - L_\tau}\,, \quad \text{and } U(1)_{L_\mu - L_\tau}\,. \tag{I-2.66}$$

However, since these are gauge symmetries, they restrict the form of κ heavily because the Weinberg-Operator $\sim \kappa_{gf}\, l^g\, l^f$ needs to be gauge-invariant. This means that, for instance, in the case of $U(1)_{L_\mu - L_\tau}$, $\kappa_{gf}\, l^g\, l^f$ is required to be invariant under the transformations

$$l_e \longrightarrow l_e\,, \tag{I-2.67}$$

$$l_\mu \longrightarrow e^{+i\,\alpha(x)\;\tilde{g}}\, l_\mu\,, \qquad \text{and} \tag{I-2.68}$$

$$l_\tau \longrightarrow e^{-i\,\alpha(x)\;\tilde{g}}\, l_\mu\,, \tag{I-2.69}$$

where $\tilde{g}$ is the gauge coupling constant. Therefore, the form of κ is restricted to

$$\kappa = \begin{pmatrix} \kappa_{11} & 0 & 0 \\ 0 & 0 & \kappa_{23} \\ 0 & \kappa_{23} & 0 \end{pmatrix}\,. \tag{I-2.70}$$

Due to this stringent form, however, this type of structure is excluded by oscillation data—see, e.g., [35] for other viable structures with vanishing entries. This means that if we assume such a flavor extension, we need to break the symmetry spontaneously to generate additional non-zero entries. Furthermore, $U(1)_{L_\mu - L_\tau}$ is frequently preferred over the other two, since its gauge boson does not couple to the electrons present in matter, which means that experimental restrictions are less severe. Furthermore, it provides an explanation for the anomalous magnetic moment of the muon, as well as potentially dark matter [11, 12]. Note, however, that if we consider, e.g., the anomalous magnetic moment of the muon, the contributions coming from the new gauge bosons are of the order of $m_\mu^2/m_{Z'}^2$. This means that the mass of the new Z' gauge boson cannot be too large if we want a sizeable contribution.

It has furthermore been shown that out of the three $U(1)_{L_\alpha - L_\beta}$ gauge extensions with either an $SU(2)_L$ singlet or doublet scalar, all but $U(1)_{L_\mu - L_\tau}$ with a singlet are excluded [10]. Note that this is not necessarily the case for extended, non-minimal models involving these gauge extensions, or with badly broken symmetries that do not constrain the structure of the neutrino mass matrix.

In most cases, these gauge symmetries are considered in the context of small symmetry breaking, and a light Z' boson—the new gauge boson coming from the $U(1)$ group. Let us consider a simple model for one such a scenario, where we add one singlet scalar to the Lagrangian in a $U(1)_{L_\mu - L_\tau}$ extension with right-handed neutrinos [11]. We define the three right-handed neutrinos N_i to have the same $U(1)_{L_\mu - L_\tau}$ charges as the leptons—0 for the first generation, +1 for the second, and -1 for the third—and the scalar S to have with $U(1)_{L_\mu - L_\tau}$ charge -1, and obtain

$$-\mathcal{L} \supset \frac{1}{2}\,\overline{N_i^C}\,M_{ij}\,N_j + Y_{\nu,ij}\,\overline{l_i}\left(\varepsilon\,\phi^*\right)N_j + Y_{N,12}\,\overline{N_1^C}\,N_2\,S + Y_{N,13}\,\overline{N_1^C}\,N_3\,S^\dagger\,. \tag{I-2.71}$$

Note that M and Y_ν have the same $U(1)_{L_\mu - L_\tau}$-symmetric structure as in eq. (I-2.70). From the seesaw mechanism, κ would also obtain this symmetric structure without symmetry breaking. After the $U(1)_{L_\mu - L_\tau}$ gauge symmetry is spontaneously broken by the vev of S, new entries are induced, and κ takes the form

$$\kappa = \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & 0 & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & 0 \end{pmatrix}\,, \tag{I-2.72}$$

where κ_{11} and κ_{23} come from M and Y_ν, whereas the newly generated κ_{12} and κ_{13} are derived from $Y_{N,12}$, $Y_{N,13}$, and the scalar vev $\langle S\rangle$. The new entries violate $U(1)_{L_\mu - L_\tau}$ by one unit, and the diagonal entries that are still zero by two units—assuming weak breaking, these are neglected. Thus, we see that via Spontaneous Symmetry Breaking (SSB) of flavor gauge extensions such as $U(1)_{L_\mu - L_\tau}$, we can generate new entries in the neutrino mass matrix, making them compatible with oscillation data, while also providing explanations for other current problems. In this work, we will discuss such gauge extensions extensively, albeit in a different realization—badly broken flavor gauge symmetries.

I-2.3.3 Neutrino Masses Beyond Tree-Level

Besides the tree-level considerations we made thus far, the question arises what happens to the neutrino mass matrix when considering loop diagrams. This leads us to the concept of running neutrino masses and mixing angles. Such contributions may noticeably change the description of neutrino masses; in particular, since we start a description at the UV scale via matching and integrating out heavy fields, but may ultimately perform measurements at comparatively low energies around the weak scale $\sim 100\,\mathrm{GeV}$. Furthermore, it may be interesting to consider the energy evolution of neutrino masses for experiments. For instance, if the production scale and the detection scale differ, then running effects may be relevant to consider. It was shown in [21] that such running effects can provide explanations for experimental anomalies. Therefore, as this work will deal with precisely the radiative effects on neutrino parameters, we will discuss core aspects of renormalization and its effect on κ in the SM in the next chapter.

CHAPTER

I-3

Renormalization Theory

After having introduced neutrino physics, let us now turn our attention to renormalization, and its effects on neutrino masses. For references of topics discussed in this chapter, see for instance [36, 30, 31, 32].

I-3.1 Introduction to Renormalization

To begin, let us first briefly discuss why we need renormalization. To that end, we recall that we start our calculation in QFT from the Lagrangian, and then perform a perturbative expansion with the interaction terms contained in it. This expansion takes the form

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \cdots, \tag{I-3.1}$$

where $\mathcal{A}$ is our total amplitude, $\mathcal{A}^{(0)}$ is the tree-level contribution, $\mathcal{A}^{(1)}$ the one-loop contribution, $\mathcal{A}^{(2)}$ the two-loop contribution, etc. At tree-level, the momenta of the virtual, intermediate particles are fixed by momentum conservation at the interaction vertices. However, starting at the one-loop level, we can fulfill momentum conservation at the vertices with arbitrary intermediate momenta. Consider, for instance, the one-loop diagram of fig. (I-3.1) that arises in, e.g., in a scalar ϕ^3 theory with the interaction Lagrangian $-\mathcal{L}_{int} = \frac{1}{3!}\, g\, \phi^3$. We denote the momentum of ϕ by p, and its mass by m.

We thus see from fig. (I-3.1) that we do indeed fulfill momentum conservation at both interaction vertices for any loop momentum ℓ. Therefore, we need to integrate over all possible loop momenta, since they contribute to the overall diagram. Therefore, this leads us to the loop-integral over the intermediate particles' propagators,

$$\mathcal{A}^{(1)} \sim g^2 \int \frac{\mathrm{d}^4\ell}{(2\pi^4)} \, \frac{i}{(p+l)^2 - m^2} \, \frac{i}{\ell^2 - m^2}. \tag{I-3.2}$$

However, if we now inspect this integral for large loop momenta $\ell \longrightarrow \infty$, we see by power-counting that this integral is *divergent*. To see this, we put a cut-off Λ on the absolute value of

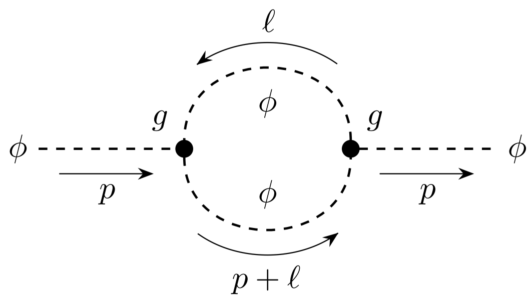


Figure I-3.1.: One-loop diagram in a scalar ϕ^3 theory; ℓ is the d-dimensional loop momentum

ℓ, and obtain

$$\int \mathrm{d}^4\ell \, \frac{1}{(p+l)^2 - m^2} \, \frac{1}{\ell^2 - m^2} \xrightarrow{\ell \longrightarrow \infty} \int \frac{\mathrm{d}^4\ell}{l^4} = \Omega_4 \int^{\Lambda} \underbrace{\frac{\mathrm{d}\ell \, \ell^3}{\ell^4}}_{= \mathrm{d}\ell/\ell} \sim \ln \Lambda \, . \tag{I-3.3}$$

Here, we have defined the total solid angle in four dimensions as Ω_4. As we see from eq. (I-3.3), the loop-integral diverges logarithmically as we send $\Lambda \longrightarrow \infty$. In general, we can perform power-counting to see whether a loop-diagram is divergent; we do this by taking the limit of large loop momenta, and counting the total number of powers it appears with. The integration measure $\mathrm{d}^4\ell$ always contributes four powers, and additional powers come from propagators, and potentially some momentum-dependent interaction vertices. In particular,

- scalar and vector propagators contribute as $\frac{1}{\ell^2}$, and
- fermion propagators contribute as $1/\ell$.

By counting powers of ℓ, we thus obtain the superficial degree of divergence,

$$\Delta = \text{powers of } \ell \text{ in numerator} - \text{powers of } \ell \text{ in denominator} \, , \tag{I-3.4}$$

and if $\Delta < 0$, the diagram is finite. If, however, $\Delta \geq 0$ it is *divergent.* In the case of eq. (I-3.3), we have $\Delta = 0$, which is called logarithmically divergent; we call $\Delta = 1$ linearly divergent, $\Delta = 2$ quadratically divergent, etc.

Thus, we have observed that loop diagrams can be divergent if their superficial degree of divergence is greater or equal to zero. However, physically, we cannot have such infinite contributions. Therefore, to calculate loop corrections to physical processes and observables, we need some scheme to make sense of and deal with these divergences. This leads us to renormalization.

I-3.2 The Renormalization Paradigm

I-3.2.1 Renormalization Constants and Counterterms

Now that we know that loop diagrams may be divergent, we need a way to deal with these divergences. Therefore, let us now introduce the concepts of renormalized perturbation theory, and in particular dimensional regularization.

In renormalized perturbation theory, we re-express the fields, couplings, masses, etc. of the Lagrangian in terms of new, *renormalized* quantities. The aim of this is to make use of these renormalized fields and the renormalization constants they appear with to cancel the divergences order-by-order in the loop expansion. For instance, we express a bare field ϕ_B—the unrenormalized one—in terms of the renormalized field ϕ by writing

$$\phi_B = Z_\phi^{1/2}\,\phi\,, \tag{I-3.5}$$

where Z_ϕ is the field renormalization constant of ϕ. Note that the power of $1/2$ is chosen such that the kinetic term $\sim \phi_B^2$ contains Z_ϕ to the power of one. We now divide Z_ϕ up into a tree-level part, and a loop part, $Z_\phi = 1 + \delta Z_\phi$. The tree level part is given by unity, as we do not need renormalization at tree-level. Thus, we obtain

$$\phi_B = Z_\phi^{1/2}\,\phi = \phi + \frac{1}{2}\,\delta Z_\phi\,\phi. \tag{I-3.6}$$

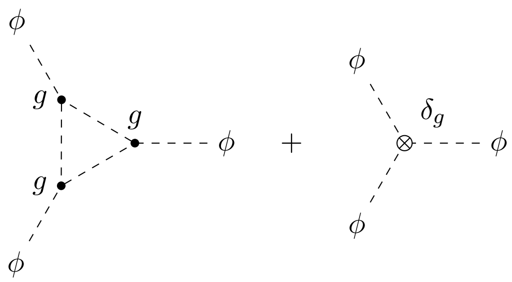
If we now express all fields, couplings, etc. in terms of these renormalized quantities, we will obtain so-called counterterms, which we will then use to cancel the divergences arising in loop-integrals. Since the δZ come from loop corrections, we assume them to be small, and only expand up to linear order in them. Note that defining the renormalized quantities by multiplying with a renormalization constant is called multiplicative renormalization; we may just as well use additive renormalization by instead defining $\delta\phi \equiv \delta Z_\phi\,\phi$. Thus, for instance, we obtain for the ϕ^3 theory, where we renormalize the coupling additively,

$$\frac{1}{3!}\,g_B\,\phi_B^3 = \frac{1}{3!}\,(g+\delta g)\,(\phi + \frac{1}{2}\,\delta Z_\phi\,\phi)^3 \tag{I-3.7}$$

$$= \frac{1}{3!}\,g\,\phi^3 + \frac{1}{3!}\,\delta g\,\phi^3 + \frac{3}{2}\cdot\frac{1}{3!}\,g\,\delta Z_\phi\,\phi^3 + \mathcal{O}(\delta g^2,\,\delta Z_\phi^2,\,\delta g\,\delta Z_\phi) \tag{I-3.8}$$

$$= \underbrace{\frac{1}{3!}\,g\,\phi^3}_{=\mathcal{L}_{int,\,ren}} + \underbrace{\frac{1}{3!}\,\delta_g\,\phi^3}_{=\mathcal{C}_g} + \mathcal{O}(\delta^2)\,. \tag{I-3.9}$$

Here, we have defined $\delta_g = \delta g + \frac{3}{2}\, g\, \delta Z_\phi$. The result we obtained in eq. (I-3.9) contains two terms; the renormalized interaction Lagrangian $\mathcal{L}_{int,\,ren}$, which has the same form as the original Lagrangian but in terms of renormalized fields, and the counterterm Lagrangian $\mathcal{C}_g$, which is linear in δ_g. As we see, the external fields in the counterterm Lagrangian are the same as in the renormalized one. This means that we can use the counterterm to cancel the divergences that arise in one-loop diagrams with three external ϕ fields. We show an example of this in eq. (I-3.10).



$$\stackrel{!}{=} \text{ finite} \qquad \text{(I-3.10)}$$

Therefore, we define the renormalization constants such that they absorb and cancel the divergences arising from the loop diagrams, rendering their contributions finite. A theory is called *renormalizable* is we can cancel all divergences—to any loop order—via a finite number of counterterms, and nonrenormalizable otherwise. As already mentioned in the previous chapter, it can be shown that all theories with couplings that have non-negative mass dimension are renormalizable, whereas those with coupling with negative mass dimension are nonrenormalizable. Note that there is one exception to this; namely, theories of non-abelian massive vector bosons. Due to a badly divergent part in their propagator, these are renormalizable if and only if they get their mass via the Higgs mechanism—i.e., via spontaneous breaking of a gauge symmetry [37].

The cancelation as we see it in eq. (I-3.10) happens in the context of vertex corrections—i.e., with interaction vertices. However, we also need to consider contributions coming from field renormalization and mass renormalization. These come from renormalizing diagrams such as the one in fig. (I-3.1). Here, the counterterm arises from the kinetic and mass terms of the fields:

$$\mathcal{L}_{kin+m} = \frac{1}{2}\, \partial^\mu \phi_B\, \partial_\mu \phi_B - \frac{1}{2}\, m_B^2\, \phi_B^2 \qquad \text{(I-3.11)}$$

$$\stackrel{m_B^2 = m^2 + \delta m^2}{=} \frac{1}{2}\, \partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2}\, m^2\, \phi^2 + \frac{1}{2}\, \delta Z_\phi\, \partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2}\, \underbrace{\left(\delta m^2 + m^2\, \delta Z_\phi\right)}_{\equiv\, \delta_{m^2}}\, \phi^2\,. \qquad \text{(I-3.12)}$$

We see here that δZ_ϕ will cancel divergences $\sim p^2$, as it appears together with derivatives. δ_{m^2} on the other hand, will cancel divergences $\sim m^2$, i.e., momentum independent ones. Note that the choice $m_B^2 = m^2 + \delta m^2$ is not unique, and it is also common to choose $m_B^2 = Z_\phi\,(m^2 + \delta m^2)$ such that $\delta_{m^2} = \delta m^2$.

Now that we know the general idea of renormalization, let us discuss how to do the calculation in practice. Namely, while we can define the counterterms such that they cancel the divergences arising in loop diagrams, we still need to extract these divergences somehow. While there are multiple schemes to do this, we will mention only some of them, and then focus on dimensional regularization, as this is one of the most common ones, and the one we use in this work.

I-3.2.2 Renormalization Schemes

The most straightforward renormalization scheme is to use a UV cutoff, as we have also seen before. Here, we cut off the momentum integration at some scale Λ, compute the diagram with this cutoff, and then cancel the arising divergence in Λ via the counterterms. For instance, we calculate a one-loop diagram that depends on some momenta p_i and masses m_i

$$\text{const.} \cdot \int \mathrm{d}^4\ell\, f(\ell,\, p_i,\, m_i) = \text{const.} \cdot \int_0^\Lambda \mathrm{d}\ell\, \ell^3 \int \mathrm{d}\Omega_4\, f(\ell,\, p_i,\, m_i) \tag{I-3.13}$$

$$= \text{const.} \cdot g(\Lambda,\, p_i,\, m_i)\Big|_{\text{div}} + \text{const.} \cdot g(\Lambda,\, p_i,\, m_i)\Big|_{\text{fin}}\,, \tag{I-3.14}$$

where f and g are some functions, and div and fin denote the divergent and finite part as $\Lambda \longrightarrow \infty$. Next, we can calculate the counterterm diagram—which is tree-level for one-loop divergences—and absorb the divergent part of g in the corresponding renormalization constant, such that only the finite part remains. Let us emphasize at this point that the exact definition of the renormalization constant is not unique. Namely, we may also absorb some finite part, which would thus change the remaining one-loop contribution. Thus, it is necessary to define renormalization conditions, and particular schemes, to ensure that the results are comparable and consistent.

Another common renormalization scheme is the so-called Pauli-Villars scheme. In this scheme, we replace propagators with the difference of the original propagator, and the propagator of some heavy particle. For instance, we would replace for a massless propagator

$$\frac{1}{\ell^2 + i\,\epsilon} \longrightarrow \frac{1}{\ell^2 + i\,\epsilon} - \frac{1}{\ell^2 - \Lambda + i\,\epsilon}\,, \tag{I-3.15}$$

where we explicitly included the $i\,\epsilon$ that is added to avoid running into a pole at $\ell^2 = 0$. We now see that in the limit $\ell \longrightarrow \infty$, the propagators cancel, and so we can calculate the resulting loop-integral. The underlying divergence is then contained in the limit of taking Λ to infinity, from which we would recover our original theory. Furthermore, we note that the small ℓ region is unaffected because the large Λ suppresses the contribution from the second term.

Let us note one important caveat of the UV cutoff and Pauli-Villars regularization schemes. Namely, they are not, in general, gauge-invariant. In particular, we cannot use them gauge-covariantly in non-abelian gauge theories in their standard form. Therefore, we will use a different renormalization scheme, dimensional regularization.

In the previous regularization methods we discussed, we calculated the loop-integrals in four dimensions and isolated the divergences using, e.g., a UV cutoff. In dimensional regularization, however, we calculate the integrals in a general d dimensions, where the integrals are finite, and then analytically continue the result to four dimensions. In practice, we thus isolate the divergences in the form of poles in $1/(4-d)$. Usually, we write d as

$$d = 4 - 2\varepsilon \,, \tag{I-3.16}$$

or $d = 4 - \varepsilon$, depending on the convention. In our convention, we thus obtain poles of the form

$$\frac{1}{2\varepsilon} \,. \tag{I-3.17}$$

One salient aspect we need to consider, is that by going to a general d dimensions, the mass dimensions of fields and couplings change. This is because the action is required to have mass dimension zero, and

$$0 \overset{!}{=} [S] = \left[\int \mathrm{d}^d x \, \mathcal{L} \right] = -d + [\mathcal{L}] \tag{I-3.18}$$

$$\implies [\mathcal{L}] = 0 \,. \tag{I-3.19}$$

This means that the mass dimension of the Lagrangian changes, and thus of the fields and couplings contained in it. Let us consider the ϕ^3 Lagrangian:

$$\mathcal{L} = \frac{1}{2} \, \partial^\mu \phi \, \partial^\mu \phi - \frac{1}{2} \, m^2 \, \phi^2 - \frac{1}{3!} \, g \, \phi^3 \,. \tag{I-3.20}$$

Since the derivative has mass dimension one, $[\partial] = 1$, requiring the mass dimension of $\mathcal{L}$ be d means that the mass dimension of ϕ is given by $[\phi] = (d-2)/2$. From this, we can then deduce the mass dimensions of the mass square and the coupling g. We can define a quantity D that encapsulates this scaling by

$$[\alpha]_d = [\alpha]_{d=4} + D_\alpha \cdot (4 - d) \,, \tag{I-3.21}$$

where α is some coupling, and the subscripts indicate the dimension. Thus, we obtain

$$D_\alpha = \frac{[\alpha]_d - [\alpha]_{d=4}}{4 - d} \,. \tag{I-3.22}$$

However, we would like to keep the dimensions of the couplings the same, and thus introduce the so-called *renormalization scale* μ. Hence, we write

$$\alpha \xrightarrow{d \text{ dimensions}} \mu^{D_\alpha} \, \alpha \tag{I-3.23}$$

In practice, we wish to use the so-called modified Minimal Subtraction, or $\overline{\text{MS}}$, scheme. Minimal subtraction means that we only absorb the $1/2\varepsilon$ poles in the renormalization constants. In the modified scheme, we cancel some additional constants that we would have to always carry with us in the expressions otherwise. This cancelation we can achieve by defining

$$\tilde{\mu} \equiv \mu \sqrt{\frac{e^{\gamma_E}}{4\pi}}\,, \tag{I-3.24}$$

where γ_E is the Euler-Mascheroni constant.
A very useful formula is to calculate one-loop integrals is given by

$$A(n,\,\Delta) \equiv \tilde{\mu}^{4-d} \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\, \frac{1}{(\ell^2-\Delta)^n} = \frac{i}{16\pi^2}\, \frac{\Gamma\Big(n-\frac{d}{2}\Big)}{\Gamma(n)} \left(\frac{\Delta}{4\pi\,\tilde{\mu}^2}\right)^{\frac{d}{2}-2} (-\Delta)^{2-n}\,, \tag{I-3.25}$$

where Γ is the gamma function. When calculating one-loop diagrams by hand in this work, we used this formula amply. Note that we also checked the results using the program Mathematica; we will comment more on this later on in this thesis. Since the derivation of this formula is rather lengthy and frequently covered in textbooks, we will not derive it here—see, e.g., [36] for a full derivation. It is, however, instructive for us at this point to consider, for instance, the case of $n = 2$, and expand this formula for $d = 4 - 2\varepsilon \longrightarrow 4$:

$$A(1,\,\Delta) = \frac{i}{16\pi^2}\, \frac{\Gamma(\varepsilon)}{\Gamma(2)} \left(\frac{\Delta}{4\pi\,\tilde{\mu}^2}\right)^{-\varepsilon} (-\Delta)^0 \tag{I-3.26}$$

$$= \frac{i}{16\pi^2}\, \frac{1}{\varepsilon}\, \Gamma(1+\varepsilon)\, \exp\left(-\,\varepsilon\, \frac{\Delta}{\mu^2\, e^{\gamma_E}}\right) \tag{I-3.27}$$

$$= \frac{i}{16\pi^2} \left(\frac{1}{\varepsilon} - \ln \frac{\Delta}{\mu^2} + \mathcal{O}(\varepsilon)\right). \tag{I-3.28}$$

We have thus isolated the divergence in the form of an explicit pole in $1/\varepsilon$, and obtained the finite contribution given by $\sim \ln \Delta/\mu^2$! The parts of order ε vanish as $\varepsilon \longrightarrow 0$. We thus see two important points:

- we can now cancel the pole in $1/\varepsilon$ via the renormalization constants, and
- the finite contribution depends on the renormalization scale.

The first point shows how we can perform the cancelation of UV divergences in practice; we expand the result of the loop-integral for small ε, cancel the divergence using counterterms, and then take the limit $\varepsilon \longrightarrow 0$. The second point seemingly tells us that physical results depend on the renormalization scale. However, since this is just an artifact of the way we perform our calculations, this cannot be true. One important point here is the fact that the physical observable *will not* depend on the renormalization scale; while a functional expression for some quantities may appear to be dependent on μ, implicit dependencies will cancel such that the numerical value of the observables is, in fact, independent of the renormalization scale. This also leads us to the concept of Renormalization Group Equations, which we will discuss soon.

Let us emphasize one important point when calculating renormalization constants, namely that these come from *UV divergences*. It is crucial to include *all* such divergences in the calculation of the renormalization constants, *including* those that come from scaleless, logarithmically divergent integrals. In dimensional regularization, we find that scaleless integrals vanish; i.e.,

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^m} = 0 \,. \tag{I-3.29}$$

where m is some integer. We see this, for instance, from the fact that when substituting $\ell = a \cdot \ell'$ for some a, we obtain

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^m} = a^{d-m} \int \frac{\mathrm{d}^d \ell'}{(2\pi)^d} \frac{1}{\ell'^m} \,. \tag{I-3.30}$$

Since the integration is over the entire space, the integrals themselves are the same; therefore, this equation cannot be fulfilled for general a, unless the integral vanishes.

In the case of the logarithmically divergent $1/\ell^4$ integral, it vanishes because the UV divergence cancels with an infrared (IR) divergence that appears when $\ell \longrightarrow 0$. We can write

$$\int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^4} \sim \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}} + \text{finite} \,, \tag{I-3.31}$$

or in other words, we also need to regularize the integral in the IR. Otherwise, if we take the naive case of $\varepsilon_{UV} = \varepsilon_{IR}$, we will obtain *wrong* results for the renormalization constants. This is because the UV and IR divergences cancel, thus changing the result for the renormalization constants, as they do not absorb this divergence anymore. However, this is not correct because the UV regularization should be independent of the IR regularization. Therefore, it is crucial to keep logarithmically divergent, scaleless integrals, and absorb their UV divergences in the renormalization constants as well. In practice, this can be done by adding an IR regulator—i.e., a mass in the propagator—and then taking the regulator to zero *after* determining the renormalization constants. Note that this becomes apparent, for instance, when calculating *physical* β-functions—we will discuss these in the next section—as not paying attention to the point we raised here can lead to explicit gauge-dependence. If, however, we include the UV poles from these logarithmically diverging, scaleless integrals, the gauge-dependence cancels, as expected.

I-3.3 Renormalization Group Equations

I-3.3.1 Why We Need Renormalization Group Equations

As we have seen before, the physical quantities we calculate using some renormalization scheme always depend on some new scale—this may be the UV cutoff, the regulator mass in Pauli-Villars regularization, the renormalization scale μ in dimensional regularization etc. However, given that these scales originate from a particular calculation scheme, we know that physical quantities cannot depend on them. This will lead us to renormalization group equations (RGEs) and running couplings. Another perspective comes from the following consideration: As we have seen in eq. (I-3.28), we obtain finite contributions of the form $\sim \ln \Delta/\mu^2$; however, Δ originates from the propagators and may, e.g., be the mass of some particles relevant to our system, the center mass energy, etc. Therefore, if we now try to compute the contribution coming from loop-diagrams directly at this relevant scale, we see that these logarithms blow up and go to infinity—i.e., the perturbation expansion is not reasonably defined anymore. This is problematic because we perform measurements at such scales, so being able to calculate corrections to physical quantities there is crucial. Renormalization group equations provide a way to essentially re-sum these large logarithms and obtain a finite result at these scales via running couplings. This means that we perform the renormalization procedure at some renormalization scale μ, and then run the RGEs down to the relevant scale for our observations.

To see how these running couplings emerge, let us consider some physical observable R, which we express in terms of dimensionless quantities—this observable may, e.g., the fraction of two cross-sections. Thus, it depends on the renormalization scale μ^2 divided by some reference scale μ_0^2, and some coupling $\alpha(\mu)$ that itself depends on the renormalization scale. The reference scale may, for instance, be the scale at which we perform our measurements. Note that we could also consider masses, but for simplicity we assume a massless theory here. Following our previous line of arguments, we know that the *total* derivative of R with respect to the renormalization scale has to vanish. We can then express the total derivative in terms of partial derivatives, and obtain

$$0 \overset{!}{=} \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} R\left(\mu^2/\mu_0^2,\, \alpha(\mu)\right) = \Bigg(\mu^2 \frac{\partial}{\partial \mu^2} + \underbrace{\mu^2 \frac{\partial \alpha(\mu)}{\partial \mu^2}}_{\equiv \beta_\alpha} \frac{\partial}{\partial \alpha(\mu)}\Bigg) R\left(\mu^2/\mu_0^2,\, \alpha(\mu)\right), \tag{I-3.32}$$

where we have defined the β-function for α, β_α as the logarithmic derivative of α. Eq. (I-3.32) is the RGE for R. To solve this equation, we essentially reparametrize our description, and put all the scale dependence in to the *running coupling*. We define

$$t = \ln \frac{\mu^2}{\mu_0^2}\,, \tag{I-3.33}$$

and thus solve the RGE of eq. (I-3.32) by

$$R\big(1, \alpha(t)\big)\,, \tag{I-3.34}$$

where now all the scale dependence is reabsorbed in the coupling. Hence, we get the RGE for α,

$$\beta_\alpha = \frac{\mathrm{d}\alpha}{\mathrm{d}t}\,, \tag{I-3.35}$$

the solution of which is the *running coupling* $\alpha(t)$. Thus, we can renormalize α at one scale, and then run the RGE to some other scale to obtain the value of α there. In practice, this means solving the differential equation of eq. (I-3.35). The question arises, however, of how we can obtain the β_α-function. To see this, we consider the bare coupling α_B, which by definition does not depend on the renormalization scale. We then express α_B in terms of the renormalized coupling and renormalization constants as some function $f(\alpha,\, \delta\alpha,\, Z_{\phi_i}$, where ϕ_i are some fields that couple via α. Taking the total derivative with respect to the renormalization scale, we obtain

$$0 \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\alpha_B = \frac{\mathrm{d}}{\mathrm{d}t} f\big(\alpha,\, Z_\alpha,\, Z_{\phi_i}\big) \supset \beta_\alpha\,, \tag{I-3.36}$$

where we observe that the derivative will act on the renormalized coupling α, thus giving us β_α. We can then invert this relation and thus obtain an expression for β_α in terms of the renormalization constants,

$$\beta_\alpha = g(\alpha,\, \delta Z_\alpha,\, \delta Z_{\phi_i})\,, \tag{I-3.37}$$

where we dropped the unity part of the renormalization constants. This means that, ultimately, we obtain β_α from loop-diagrams. This is not surprising, given that RGEs originate from a regularization-scale-independence, and the regularization scale appears only at the loop level.

I-3.3.2 Renormalization Group Equations from Tensorial Counterterms

Now that we have seen the origin of RGEs and why they are important, let us discuss how to calculate β-functions from tensorial counterterms—this subsection is based on [19, 34].

We start from the Lagrangian term of some quantity Q with some fields $\phi_{i/j}$,

$$\mathcal{L} \sim \left(\prod_{i\in I} \phi_i^{2n_i}\right) Q \left(\prod_{j\in J} \phi_j^{2n_j}\right) \tag{I-3.38}$$

where we parametrize the power the fields appear with by the half-integers $n_{i/j}$. We have also defined $I = \{1, ..., M\}$ with M the number of fields appearing to the left of Q, and $J = \{M+1, ..., N\}$, where N is the total number of fields appearing together with Q. Thus, we can express the bare quantity by

$$Q_B = \left(\prod_{i \in I} Z_{\phi_i}^{n_i} \right) \, [Q + \delta Q] \; \tilde{\mu}^{D_Q\,(4-d)} \left(\prod_{j \in J} Z_{\phi_j}^{n_j} \right) . \tag{I-3.39}$$

The renormalization constants are functions of Q and some other variables V_A, i.e.,

$$\delta Q = \delta Q\big(Q, \, \{V_A\}\big) \, , \tag{I-3.40}$$

$$Z_{\phi_i} = Z_{\phi_i}\big(Q, \, \{V_A\}\big) \, , \quad \text{with}\, i \in \{1, ..., \, N\} \, . \tag{I-3.41}$$

The V_A may, for instance, be some coupling constants, masses, etc. Since we would now like to derive an expression for the β-function of Q,

$$\beta_Q = \frac{\mathrm{d}Q}{\mathrm{d}t} \, , \tag{I-3.42}$$

we take the total derivative of the *bare* quantity with respect to the renormalization scale, and express it in terms of partial derivatives of the renormalization constants, as well as β-function. As discussed previously, we know that this derivative has to vanish, and we can thus derive an expression for β_Q. To simplify the notation, we define a product $\langle \cdot \, | \, \cdot \rangle$ for an arbitrary tensor F by

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \, \middle| \, y \right\rangle = \frac{\mathrm{d}F}{\mathrm{d}x} \, y \qquad \text{for scalars } x \text{ and } y, \tag{I-3.43}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \, \middle| \, y \right\rangle = \sum_m \frac{\mathrm{d}F}{\mathrm{d}x_m} \, y_m \qquad \text{for vectors } x \text{ and } y, \tag{I-3.44}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \, \middle| \, y \right\rangle = \sum_{m,\,n} \frac{\mathrm{d}F}{\mathrm{d}x_{m,\,n}} \, y_{m,\,n} \qquad \text{for matrices } x \text{ and } y, \text{ and} \tag{I-3.45}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \, \middle| \, y \right\rangle = \sum_{\substack{m_1,\, m_2,\\ m_3, \dots}} \frac{\mathrm{d}F}{\mathrm{d}x_{m_1,\, m_2,\, m_3, \dots}} \, y_{m_1,\, m_2,\, m_3, \dots} \qquad \text{for arbitrary tensors } x \text{ and } y. \tag{I-3.46}$$

Using this notation, we obtain from eq. (I-3.39)

$$\begin{aligned}
0 \overset{!}{=} \tilde{\mu}^{-D_Q\,(4-d)} \, \mu \, \frac{\mathrm{d}}{\mathrm{d}\mu} Q = & \left(\prod_{i \in I} Z_{\phi_i}^{n_i} \right) \Bigg[\beta_Q + \left\langle \frac{\mathrm{d}\delta Q}{\mathrm{d}Q} \, \middle| \, \beta_Q \right\rangle + \sum_A \left\langle \frac{\mathrm{d}\delta Q}{\mathrm{d}V_A} \, \middle| \, \beta_{V_A} \right\rangle \; + \\
& + \; (4-d) \, D_Q \, [Q + \delta Q] \Bigg] \left(\prod_{j \in J} Z_{\phi_j}^{n_j} \right) + \\
& + \left(\prod_{i \in I} Z_{\phi_i}^{n_i} \right) \, [Q + \delta Q] \, \Bigg\{ \sum_{j \in J} \left(\prod_{j' < j} Z_{\phi_{j'}}^{n_{j'}} \right) \times
\end{aligned} \tag{I-3.47}$$

$$\times \left[\left\langle \frac{\mathrm{d}Z_{\phi_j}^{n_j}}{\mathrm{d}Q} \middle| \beta_Q \right\rangle + \sum_A \left\langle \frac{\mathrm{d}Z_{\phi_j}^{n_j}}{\mathrm{d}V_A} \middle| \beta_{V_A} \right\rangle \right] \left(\prod_{j''>j} Z_{\phi_{j''}}^{n_{j''}} \right) \Bigg\} +$$
$$+ \left\{ \sum_{i\in I} \left(\prod_{i'<i} Z_{\phi_{i'}}^{n_{i'}} \right) \left[\left\langle \frac{\mathrm{d}Z_{\phi_i}^{n_i}}{\mathrm{d}Q} \middle| \beta_Q \right\rangle + \sum_A \left\langle \frac{\mathrm{d}Z_{\phi_i}^{n_i}}{\mathrm{d}V_A} \middle| \beta_{V_A} \right\rangle \right] \times \right.$$
$$\left. \times \left(\prod_{i''>i} Z_{\phi_{i''}}^{n_{i''}} \right) \right\} [Q + \delta Q] \left(\prod_{j\in J} Z_{\phi_j}^{n_j} \right) .$$

Note that we used the fact that the renormalization constants do not depend on the renormalization scale explicitly—see, for instance, eq. (I-3.28), where μ did not appear in the divergent term, only in the finite one.

The next step is to expand the renormalization constants in terms of their poles in $1/(4-d)$:

$$\delta Q = \sum_{k\geq 1} \frac{\delta Q,k}{(4-d)^k} , \tag{I-3.48}$$

$$Z_{\phi_i} = \mathbb{1} + \sum_{k\geq 1} \frac{\delta Z_{\phi_i},k}{(4-d)^k} = \mathbb{1} + \delta Z_{\phi_i} . \tag{I-3.49}$$

While these renormalization functions diverge as $d \longrightarrow 4$, we know that the β-functions do not. Since they are physical objects, they will remain finite as we take this limit. Therefore, we can expand the β-functions in powers of $(4-d)$ up to some integer n, from which we obtain

$$\beta_Q = \beta_Q^{(0)} + (4-d)\, \beta_Q^{(1)} + \cdots + (4-d)^n\, \beta_Q^{(n)} , \tag{I-3.50}$$

$$\beta_{V_A} = \beta_{V_A}^{(0)} + (4-d)\, \beta_{V_A}^{(1)} + \cdots + (4-d)^n\, \beta_{V_A}^{(n)} . \tag{I-3.51}$$

If we now consider the derivatives of the field renormalization constants, we see that these are at least of the order $1/(4-d)$,

$$\frac{\mathrm{d}}{\mathrm{d}\{Q,V_A\}} Z_{\phi_i}^{n_i} = \frac{\mathrm{d}}{\mathrm{d}\{Q,V_A\}} \big(\mathbb{1} + \delta Z_{\phi_i}\big)^{n_i} = n_i \Big[\mathbb{1} + \mathcal{O}\big(1/(4-d)\big)\Big]^{n_i-1} \frac{\mathrm{d}\delta Z_{phi_i}}{\mathrm{d}\{Q,V_A\}} \tag{I-3.52}$$

$$= \mathcal{O}\big(1/(4-d)\big) . \tag{I-3.53}$$

We can now use this fact to show iteratively that almost all higher-order β-functions $\beta_Q^{(k)}$ vanish. Starting with $\beta_Q^{(n)}$, we see that we get a contribution in the first line of eq. (I-3.47) form β_Q; however, due to the renormalization constants and their derivatives being at least of the order $1/(4-d)$, they cancel at least one power of $(4-d)^n$ in the other terms. Thus, at order $(4-d)$, we are left with

$$0 = (4-d)^n\, \beta_Q^{(n)} + \mathcal{O}\big((4-d)^{n-1}\big) . \tag{I-3.54}$$

Therefore, $\beta_Q^{(n)}$ vanishes. We can now repeat the same reasoning for $n-1$, $n-2$, etc., iteratively showing that they all vanish. The first nonzero contribution comes from $\beta_Q^{(1)}$. Due to the $(4-d)\,D_Q\,Q$ term in the second line of eq. (I-3.47), we now have a second term that needs to cancel with $\beta_Q^{(1)}$, and thus obtain

$$\beta_Q^{(k)} = 0\,, \quad \forall\, k\, \in \{2, ..., n\}\,, \tag{I-3.55}$$

$$\beta_Q^{(1)} = -(4-d)\, D_Q\, Q\,. \tag{I-3.56}$$

We can now repeat the same considerations as for Q and β_Q for analogous formulae for V_A and β_{V_A}. From this, we thus obtain

$$\beta_{V_A}^{(k)} = 0\,, \quad \forall\, k\, \in \{2, ..., n\}\,, \tag{I-3.57}$$

$$\beta_{V_A}^{(1)} = -(4-d)\, D_{V_A}\, V_A\,. \tag{I-3.58}$$

While these terms vanish in the limit $d \longrightarrow 4$, we need them to obtain $\beta_Q^{(0)}$ from eq. (I-3.47). Namely, we plug in the expressions we have obtained for the β-functions to cancel the linear poles in $1/(4-d)$ of δQ and δZ_{ϕ_i}. Thus, we finally obtain

$$\begin{aligned}\beta_Q^{(0)} \;=\; & \left[D_Q \left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle - D_Q\, \delta Q_{,1} \right] + \\ & + Q \cdot \sum_{j\in J} n_j \left[D_Q \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] + \\ & + \sum_{i\in I} n_i \left[D_Q \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] \cdot Q\,.\end{aligned} \tag{I-3.59}$$

We also note that complex quantities contain two degrees of freedom, this means that we need to consider Q^* and V_A^* as independent variables if they are complex. In other words, if the quantities are complex, we also need to take derivatives etc. with respect to their complex conjugates. Furthermore, this formula can be applied to multiplicative and additive renormalization.

Thus, we can now calculate the β-function for tensorial quantities, and including tensorial counterterms. In particular, we use this because we are ultimately interested in renormalizing the Weinberg-Operator, which is a matrix in flavor-space. Let us thus now move on to the renormalization of κ in the Standard Model.

I-3.4 Renormalizing the Weinberg-Operator in the Standard Model

The RGEs for the Weinberg-Operator have been calculated first in [13, 14] and later revised in [18, 19]. Since we will employ a similar formalism in this work, we will follow [19, 34] for the derivation.

We know from eq. (I-3.59) that the renormalization constant of the coupling itself, and the wave function renormalization constants of the fields contained in the vertex contribute to the RGEs of an operator. If we now consider the bare coupling for κ, we find from the form of the Weinberg-Operator that

$$\kappa_B = Z_\phi^{-\frac{1}{2}} \left(Z_l^T\right)^{-\frac{1}{2}} \left(\kappa + \delta\kappa\right) Z_l^{-\frac{1}{2}} Z_\phi^{-\frac{1}{2}} \,. \tag{I-3.60}$$

Thus, we need to consider one-loop diagrams that give contributions to $\delta\kappa$, δZ_l, and δZ_ϕ. As discussed in previous sections, we obtain δZ_l and δZ_ϕ from diagrams such as in fig. (I-3.1)—so-called self-energy diagrams. We schematically show the corresponding diagrams in fig. (I-3.2) and (I-3.3). On the other hand, contributions to $\delta\kappa$ arise from vertex corrections, which we show schematically in fig. (I-3.4). Note that we drop $SU(2)_L$ indices for clarity, and in the case of the vertex corrections, we draw each type of diagram just once, and do not repeat the ones that involve the same vertices but different combinations of the external fields. For instance, if we can have the same type of interaction with the left lepton-doublet leg, and the right one, we only draw one of the diagrams, as the other one is similar in nature.

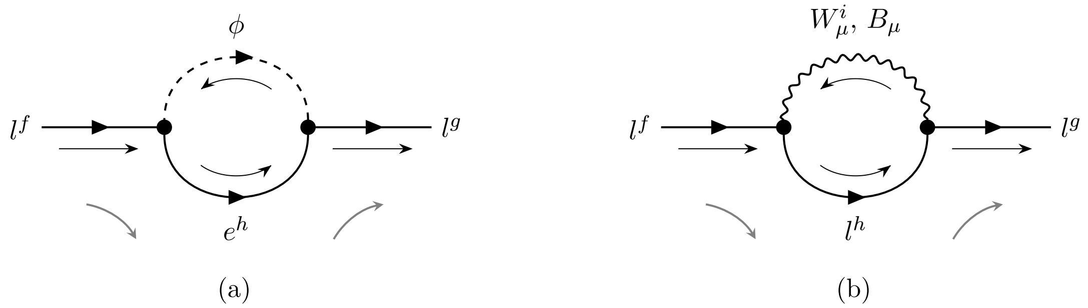


Figure I-3.2.: Contributions to the field renormalization of the lepton-doublet, δZ_l; fig. (I-3.2a) shows the contribution coming from the Higgs-doublet and the right-handed, charged lepton, fig. (I-3.2b) the ones coming from the electroweak gauge bosons, W^i_μ for $SU(2)_L$, and B_μ for $U(1)_Y$; the gray arrows denote fermion flow

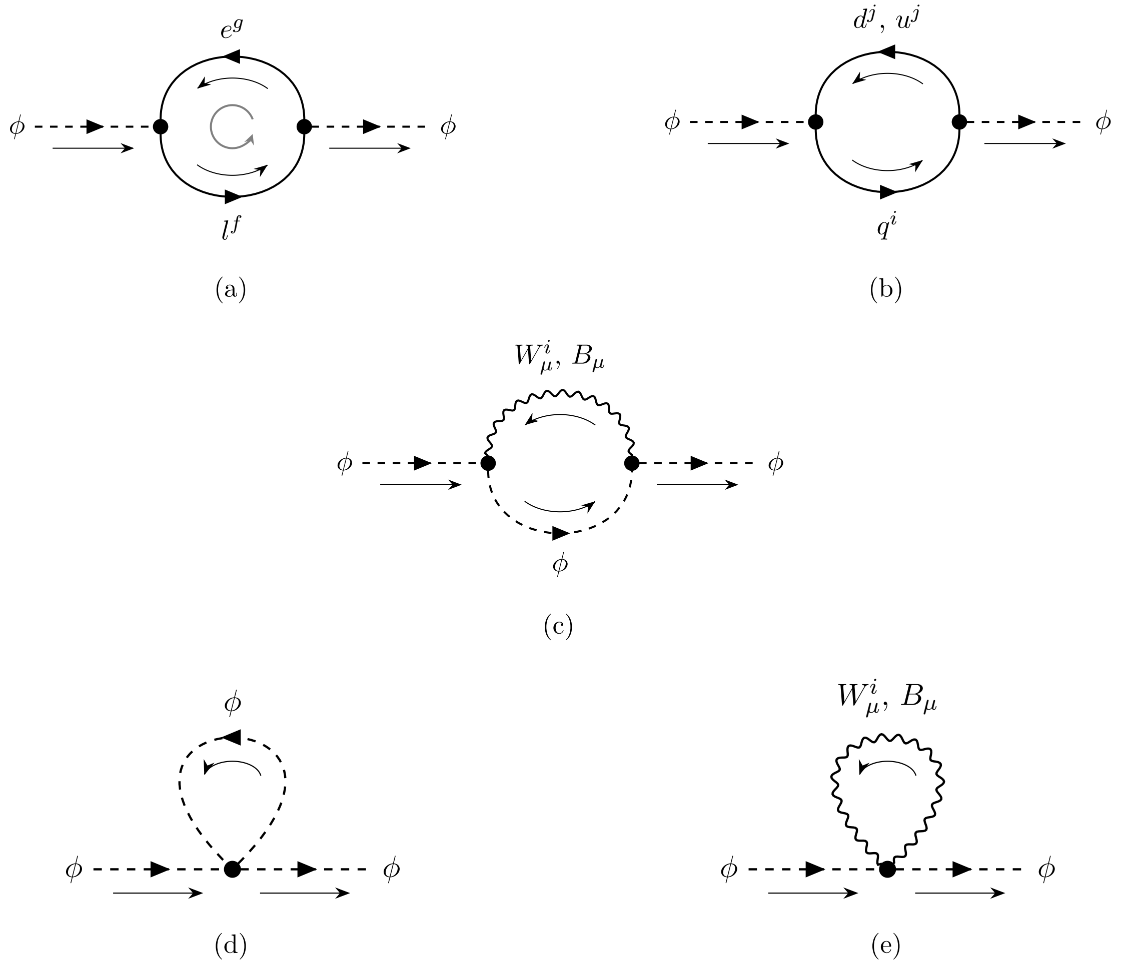


Figure I-3.3.: Self-energy diagrams of the Higgs-doublet; fig. (I-3.3a) shows the contribution coming from the lepton-doublet and the right-handed, charged lepton, fig. (I-3.3b) the ones coming from the left-handed quark-doublet and the right-handed up- and down-type quarks, fig. (I-3.3c) the one from the Higgs-doublet and electroweak gauge bosons, fig. (I-3.3d) from only the Higgs-doublet, and fig. (I-3.3e) from only the gauge bosons; the gray arrow denotes fermion flow

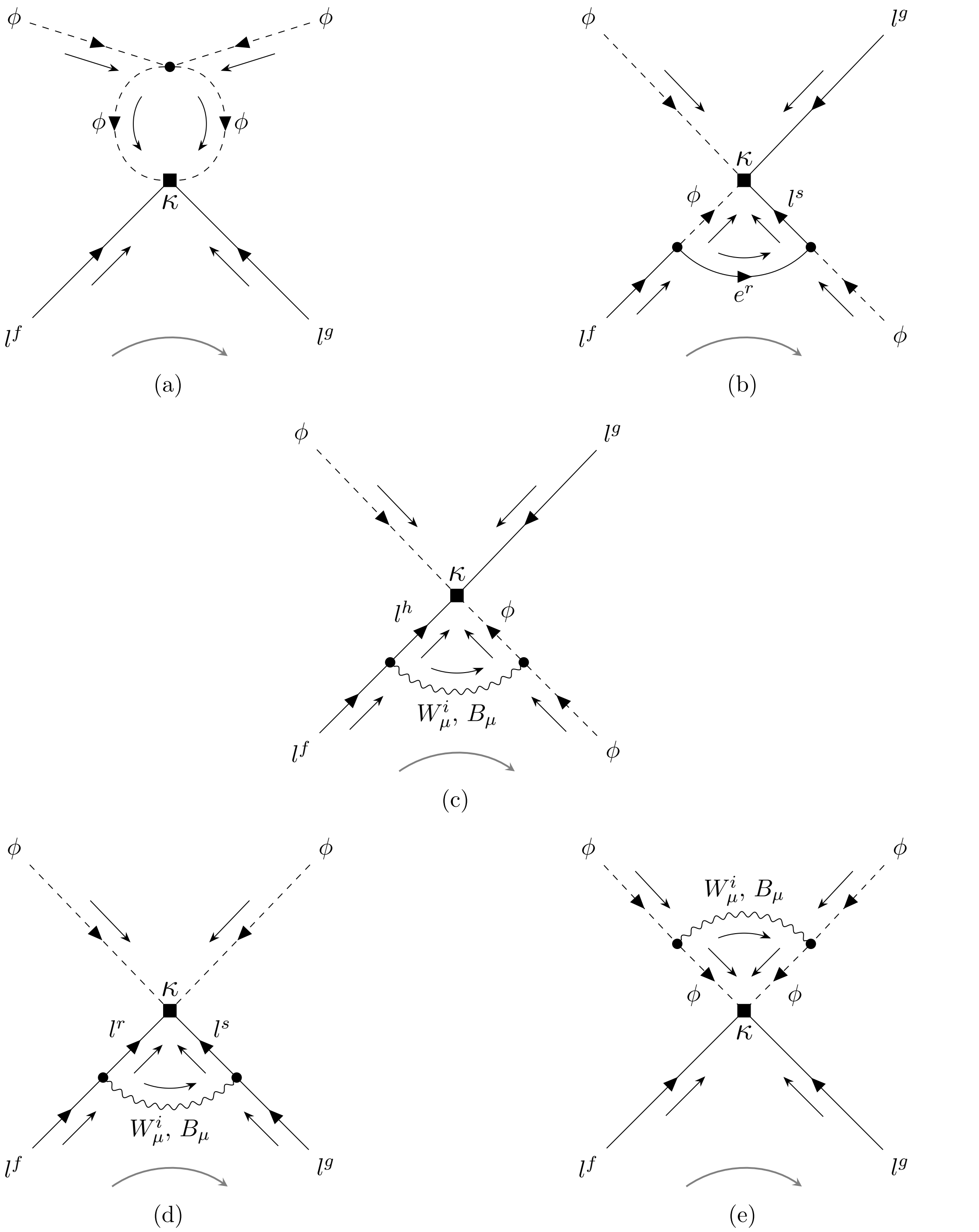


Figure I-3.4.: Vertex correction diagrams for κ; fig. (I-3.4a) shows the contribution coming from the quartic Higgs self-coupling, fig. (I-3.4b) the one coming from the right-handed, charged leptons, fig. (I-3.4c) the one from the electroweak gauge bosons being exchanged between the lepton- and Higgs-doublets, fig. (I-3.4d) from the gauge bosons being exchanged between the lepton-doublet legs, and fig. (I-3.4e) from the exchange between the Higgs-doublet legs; the gray arrows denote fermion flow

Note that since we are dealing with fermion-number-violating interactions, we need to consider how exactly to compute these, as we cannot "go against the flow" in the same way as we usually do in QFT. Therefore, we will follow the formalism of [38], wherein a "fermion flow" separate from the "fermion-number flow" is defined. We indicate fermion flow by gray arrows in diagrams, and fermion-number flow refers to the flow noted on the fermion lines themselves. For our purposes here, the formalism reduces to

- consistently fixing a fermion flow;
- going against this flow as usual;
- if fermion-number flow and fermion flow oppose each other, replacing propagators $D(p,\, m) \longrightarrow D(-p,\, m)$ and vertices $\Gamma \longrightarrow \Gamma' = C\,\Gamma^T\,C^{-1}$; and
- if the diagram does not contain fermion-number-violating interactions, choose fermion flow to be identical to fermion-number flow.

The charge-conjugation matrix C fulfills the relations

$$C^\dagger = C^{-1}\,, \qquad C^T = -C\,, \qquad C\,\Gamma\,C^{-1} = \eta_\Gamma\,\Gamma \tag{I-3.61}$$

$$\eta_\Gamma = \begin{cases} 1\,, & \text{for}\, \Gamma = \mathbb{1},\, \gamma_5,\, \gamma_\mu\,\gamma_5\,, \\ -1\,, & \text{for}\, \Gamma = \gamma_\mu,\, \sigma_{\mu\nu} = -\frac{i}{4}\left[\gamma_\mu\,,\,\gamma_\nu\right]. \end{cases} \tag{I-3.62}$$

The contributions coming from each of the diagrams are listed in [34]. The renormalization constants are then obtained by requiring that the counterterms cancel the divergences arising in the corresponding loop-diagrams. We have seen previously that wave function renormalization constants and mass renormalization constants arise from the kinetic and mass terms, respectively. In particular, for the left-handed lepton-doublet and the Higgs doublet these are given by

$$\mathcal{C}_{kin,\,l} = \delta Z_{l,\,gf}\,\overline{l^g}\,i\,\gamma^\mu\,\partial_\mu l^f\,, \tag{I-3.63}$$

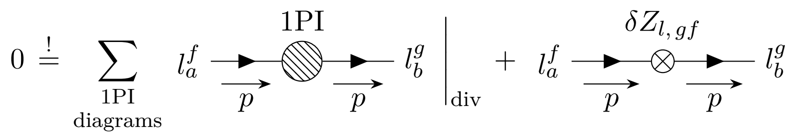
$$\mathcal{C}_{kin+m,\,\phi} = \delta Z_\phi\,\left(\partial^\mu\phi\right)^\dagger\,\partial_\mu\phi - \delta m^2\,\phi^\dagger\,\phi\,. \tag{I-3.64}$$

The counterterm for the Weinberg-Operator, on the other hand, comes from the definition of the Weinberg-Operator of eq. (I-2.51) and of the bare coupling of eq. (I-3.60). Thus, the counterterm is given by

$$\mathcal{C}_\kappa = \frac{1}{4}\,\delta\kappa_{gf}\,\overline{l_c^{g,\,C}}\,\varepsilon^{cd}\,\phi_d\,l_b^f\,\varepsilon^{ba}\,\phi_a\, + \text{h.c.}\,, \tag{I-3.65}$$

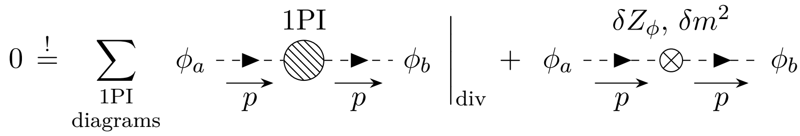
for which we observe that we obtain the same Feynman-rule as for κ by making the substitution $\kappa \longrightarrow \delta\kappa$.

Diagrammatically, we now obtain the renormalization constants by requiring



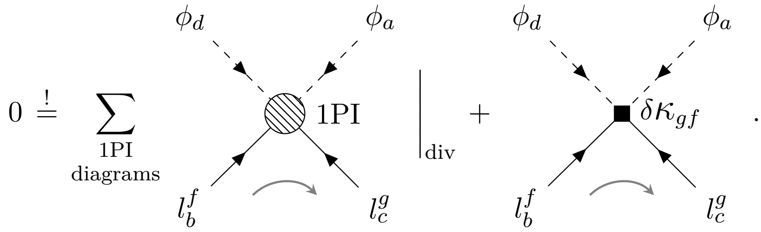
(I-3.66)

for the lepton-doublet, and



(I-3.67)

for the Higgs-doublet. Note that we have reinstated the explicit $SU(2)_L$ indices here, and the sum over one-particle-irreducible (1PI) diagrams refers to the one-loop contributions of fig. (I-3.2) and (I-3.3). Recall that 1PI diagrams are such where we cannot split the diagram by cutting a single intermediate particle's propagator. Furthermore, "div" refers to taking only the divergent parts of the respective diagrams. Similarly, we obtain $\delta\kappa$ from requiring



(I-3.68)

Now, we can calculate the divergent parts of the diagrams of fig. (I-3.2)–(I-3.4) and—if applicable—the analogous diagrams arising from the same interaction between different external fields. Then, we employ eq. (I-3.66)–(I-3.68) to obtain the renormalization constants. The results of this are given by [19]

$$\delta Z_{l,\,1} = -\frac{1}{16\pi^2}\left[Y_e^\dagger\, Y_e + \frac{1}{2}\,\xi_B\, g_1^2 + \frac{3}{2}\,\xi_W\, g_2^2\right], \tag{I-3.69}$$

$$\delta Z_{\phi,\,1} = -\frac{1}{16\pi^2}\left[2\,\mathrm{Tr}\left(Y_e^\dagger\, Y_e + 3Y_u^\dagger\, Y_u + 3Y_d^\dagger\, Y_d\right) - \frac{1}{2}\left(3-\xi_B\right)g_1^2 - \frac{3}{2}\left(3-\xi_W\right)g_2^2\right], \tag{I-3.70}$$

$$\delta\kappa_{,\,1} = -\frac{1}{16\pi^2}\Bigg[2\kappa\left(Y_e^\dagger\, Y_e\right) + 2\left(Y_e^\dagger\, Y_e\right)^T\kappa - \lambda\,\kappa + \tag{I-3.71}$$

$$- \left(\frac{3}{2} - \xi_B\right) g_1^2\,\kappa - \left(\frac{3}{2} - 3\xi_W\right) g_2^2\,\kappa\Bigg].$$

Here, Y_e, Y_u, and Y_d are the Yukawa coupling matrices of the charged leptons, up-type, and down-type quarks; ξ_B and ξ_W are the gauge parameters of $U(1)_Y$ and $SU(2)_L$; g_1 and g_2 are the $U(1)_Y$ and $SU(2)_L$ gauge couplings; and λ is the quartic Higgs self-coupling.

Now that we have the renormalization constants, we can use eq. (I-3.59) and apply it to our results above. In our case, $Q = \kappa$, and the V_A are the other quantities appearing in eq. (I-3.69)–(I-3.71). Note again that complex conjugates count as independent variables. Furthermore, we find from eq. (I-3.60) that

$$\phi_{i=1} = \phi\,, \qquad \text{(I-3.72)} \qquad\qquad n_{i=1} = -\frac{1}{2}\,, \qquad \text{(I-3.76)}$$

$$\phi_{i=2} = l\,, \qquad \text{(I-3.73)} \qquad\qquad n_{i=2} = -\frac{1}{2}\,, \qquad \text{(I-3.77)}$$

$$\phi_{j=1} = l\,, \qquad \text{(I-3.74)} \qquad\qquad n_{j=1} = -\frac{1}{2}\,, \quad \text{and} \qquad \text{(I-3.78)}$$

$$\phi_{j=2} = \phi\,; \qquad \text{(I-3.75)} \qquad\qquad n_{j=2} = -\frac{1}{2}\,, \qquad \text{(I-3.79)}$$

Thus, by taking the derivatives of the renormalization constants, we find that in the Standard Model, the β_κ-function is given by

$$\beta_\kappa \;=\; \delta\kappa_{,1} - \frac{1}{2}\left(\delta Z_{\phi,1} + \delta Z_{l,1}\right)^T \kappa - \frac{1}{2}\,\kappa\left(\delta Z_{\phi,1} + \delta Z_{l,1}\right). \qquad \text{(I-3.80)}$$

And plugging in the expressions for the renormalization constants, we finally obtain

$$\begin{aligned} 16\pi^2\,\beta_\kappa \;=\; & -\frac{3}{2}\left[\kappa\left(Y_e^\dagger Y_e\right) + \left(Y_e^\dagger Y_e\right)^T \kappa\right] + \lambda\,\kappa - 3g_2^2\,\kappa\, + \\ & + \;2\,\mathrm{Tr}\left(3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e\right)\kappa\,. \end{aligned} \qquad \text{(I-3.81)}$$

Note that even though the renormalization constants of eq. (I-3.69)–(I-3.71) are gauge-dependent, the *physical* β_κ-function is not. Since the β_κ-function determines the running of the masses, this is expected, and necessary, as physical quantities cannot depend on our gauge choice.

Following [16, 17], we can write the structure of the SM β_κ-function as

$$\beta_\kappa = \alpha\,\kappa + P^T\,\kappa + \kappa\,P\,. \qquad \text{(I-3.82)}$$

Here, we defined the flavor-space scalar α, and the flavor-space matrix P, which are given by

$$\alpha = \lambda - 3g_2^2 + 2\,\mathrm{Tr}\left(3Y_u^\dagger\, Y_u + 3Y_d^\dagger\, Y_d + Y_e^\dagger\, Y_e\right), \tag{I-3.83}$$

$$P = -\frac{3}{2}\big(Y_e^\dagger\, Y_e\big)\,. \tag{I-3.84}$$

Furthermore, this structure also holds in the Minimal Supersymmetric Standard Model, although we do not cover it here.

At this point, it would also be fitting to derive the RGEs for the eigenvalues of κ starting from eq. (I-3.82). We do this by diagonalizing κ with the leptonic mixing matrix U, as described in (I-2.64), and making an ansatz for the evolution of the mixing matrix itself. However, as we will perform this in a more general setting that includes the SM later on in this thesis, we will not do this here.

Thus, we conclude this chapter on renormalization, and with it the first, introductory part of this work. In the next part, we will apply the knowledge and principles discussed in this part to a more general class of theories, and—among other things—discover novel contributions and their implications.

PART

II

General and Flavor-Nonuniversal Renormalization of Neutrino Parameters

In the previous part we reviewed neutrino physics, renormalization, and the renormalization group equations for the effective dimension-5 Weinberg-Operator in the Standard Model. Building on this foundation, we will discuss the renormalization of neutrino parameters in general, and in particular in *flavor-nonuniversal* gauge extensions of the Standard Model. Furthermore, we will discuss the new quantum effects flavor-nonuniversal gauge extensions introduce, propose specific models realizing these effects, and present the results obtained in these models.

CHAPTER

II-1

The New Quantum Effect

To introduce the new quantum effect on neutrino parameters arising in a particular class of models, let us consider neutrino masses in the $U(1)_{L_\mu - L_\tau}$ gauge extension of the Standard Model. Since this chapter is focused on showing how the new effect arises, we will not go into much detail concerning specific calculations at this point. However, we will see that flavor-nonuniversal gauge extensions—in particular flavor gauge theories—such as $U(1)_{L_\mu - L_\tau}$ lead to a new term in the one-loop β-function for neutrino masses which is not present in the Standard Model. Furthermore, even without detailed calculations, we will understand the origin of this new term and what distinguishes our considerations here from the Standard Model, as well as the notable impact the new term has on neutrino masses.

As explained in subsection (I-2.3.2), additional flavor $U(1)$ gauge groups need to be broken spontaneously to conform with current experimental data. How specifically this is done is very model-dependent and may thus change some details of explicit calculations. However, in any flavor gauge extension, there are interactions of the new gauge bosons with the SM particles. In particular, in our $U(1)_{L_\mu - L_\tau}$ extension, the left-handed lepton doublets and right-handed charged lepton singlets are charged under this group, and thus have gauge interactions with the new Z' gauge-boson. This interaction arises as usual from the covariant derivative,

$$D_\mu = \mathbb{1}\,\partial_\mu + i\,g\,\mathbb{1}\,W^a_\mu\,T^a + i\,g'\,Y\,\mathbb{1}\,B_\mu + i\,\tilde{g}\,\tilde{Q}\,Z'_\mu\,, \tag{II-1.1}$$

where the second and third term constitute the interactions with the electroweak gauge bosons, and the last term comes from the new $U(1)_{L_\mu - L_\tau}$ gauge group, with $\tilde{g}$ being the new coupling strength. The covariant derivative in eq. (II-1.1) is written as a 3×3 matrix in lepton-flavor space, which we will often refer to simply as flavor-space. In this space, SM interactions are given by unit matrices, and the flavor gauge-interaction is characterized by the $L_\mu - L_\tau$ charge matrix $\tilde{Q}$, with

$$\tilde{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{II-1.2}$$

From the kinetic term in the Lagrangian,

$$\mathcal{L}_{kin} \supset \overline{l_f}\,(i\,\gamma^\mu\,D_\mu^{fg})\,l_g\,, \tag{II-1.3}$$

we see that the interaction with the Z'—as for all other gauge bosons as well—preserves chirality. As we will see shortly, this means that we have two possible Z'-contributions to the one-loop renormalization of the effective neutrino mass operator. Namely, we are looking for one-loop diagrams involving the Z', which renormalize the lepton-number-violating interaction

$$\mathcal{L}_{int} \supset \frac{1}{4}\,\kappa_{gf}\,\overline{l_c^{g,C}}\,\varepsilon^{cd}\,\phi_d\,l_b^f\,\varepsilon^{ba}\,\phi_a\, + \text{h.c.} \sim \kappa_{gf}\,l^g\,l^f\,\phi\,\phi\, + \text{h.c.}\,, \tag{II-1.4}$$

where we have simplified the expression and suppressed $SU(2)$-indices to emphasize the lepton-flavor structure. We show the Feynman-diagram for this interaction in fig. (II-1.1).

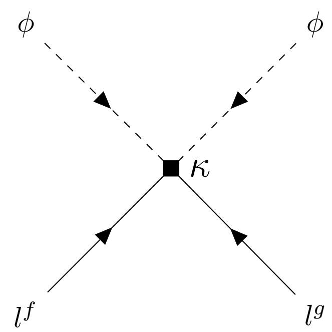


Figure II-1.1.: Feynman diagram of the Weinberg-Operator

To see how the Z' can affect the renormalization of the Weinberg-Operator at the one-loop level, let us recall eq. (I-3.80),

$$\beta_\kappa = \delta\kappa_{,1} - \frac{1}{2}\,(\delta Z_{\phi,1} + \delta Z_{l,1})^T\,\kappa - \frac{1}{2}\,\kappa\,(\delta Z_{\phi,1} + \delta Z_{l,1})\,. \tag{II-1.5}$$

We see that possible contributions at the one-loop level come from the wave function renormalization of the lepton doublets, and the vertex corrections—the wave function renormalization of the Higgs doublet does not contribute via the Z' since the doublet is not charged under $U(1)_{L_\mu - L_\tau}$. Let us first consider the wave function renormalization. The one-loop contribution of the Z' is given by the bubble diagram visible in fig. (II-1.2); it arises via two insertions of the $\overline{l^g}\,l^g\,Z'$-coupling from the covariant derivative.

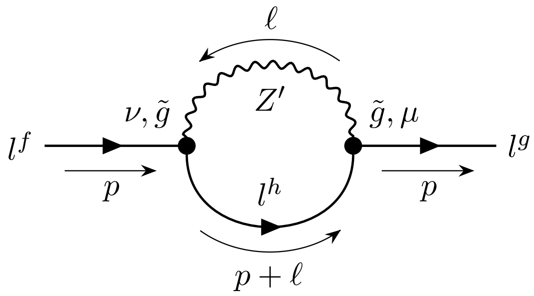
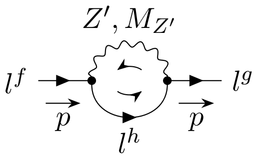


Figure II-1.2.: Wave function renormalization diagram for the lepton doublet due to the Z' gauge boson; ℓ is the d-dimensional loop momentum

We compute this diagram using the Feynman-rules listed in appendix and find,



$$= (-i\,\tilde{g})^2\,\tilde{Q}_{gh}\,\tilde{Q}_{hf}\cdot\mathcal{I}_{\delta Z_l}(p, M_{Z'}, d)\,, \tag{II-1.6}$$

where $\mathcal{I}_{\delta Z_l}(p, M_{Z'}, d)$ is the d-dimensional loop-integral over the loop momentum ℓ. This integral contains the propagators and vertex Feynman-rules, including γ-matrices. For this discussion here, we have factored out the lepton-flavor dependent part and gauge coupling prefactor. We could now perform the integration over the loop momentum in dimensional regularization and cancel the emerging UV-divergence in $\mathcal{I}_{\delta Z_l}(p, M_{Z'}, d)$ by the counterterm $\sim \delta Z_l$. Doing this, we would find for the wave function renormalization in the $\overline{\text{MS}}$-scheme

$$\delta Z_{l,\,1}\Big|_{U(1)_{L_\mu - L_\tau}} \propto \frac{\tilde{g}^2}{16\pi^2}\left(\tilde{Q}\right)^2 = \frac{\tilde{g}^2}{16\pi^2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\,. \tag{II-1.7}$$

Therefore, due to the interaction with the Z', the beta function of the Weinberg-Operator gains the additional term

$$\begin{aligned}\beta_{\kappa} &\supset \left[\delta Z_{l,\,1}\Big|_{U(1)_{L_\mu - L_\tau}}\right]^T \kappa + \kappa\,\left[\delta Z_{l,\,1}\Big|_{U(1)_{L_\mu - L_\tau}}\right]\\ &\propto \left[\frac{\tilde{g}^2}{16\pi^2}\left(\tilde{Q}\right)^2\right]^T \kappa + \kappa\,\left[\frac{\tilde{g}^2}{16\pi^2}\left(\tilde{Q}\right)^2\right]\,.\end{aligned} \tag{II-1.8}$$

Now recall the β-function for the Weinberg-Operator in the SM of eq. (I-3.82),

$$\beta_{\kappa} = \alpha\,\kappa + P^T\,\kappa + \kappa\,P\,. \tag{II-1.9}$$

By comparing eq. (II-1.9) with eq. (II-1.8), we see that the new contribution fits perfectly well into the RGE structure observed within the SM; we simply have to add the new term to the definition of the P matrix,

$$P_{\text{SM}+U(1)_{L_\mu-L_\tau}} = P_{\text{SM}} + P_{U(1)_{L_\mu-L_\tau}}\,, \qquad \text{(II-1.10)}$$

We can understand that this Z' contribution fits the same structure as the SM by realizing that, diagrammatically, the wave function renormalization has two conjugate contributions to fig. (II-1.1). Namely, we can have a Z' loop on either lepton leg entering the interaction, as displayed in fig. (II-1.3).

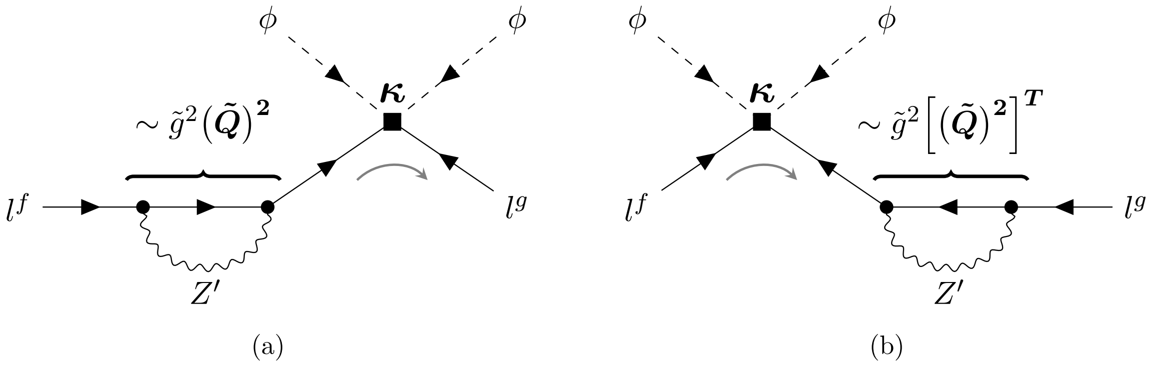


Figure II-1.3.: Structure of the Z' contribution to the Weinberg-Operator via field renormalization of lepton-doublets; the gray arrow denotes fermion flow; by going against fermion flow, fig. (II-1.3a) leads to a β_κ structure $\sim \kappa\,(\tilde{Q})^2 \subset \kappa\,P$, fig. (II-1.3b) to $\sim \left[(\tilde{Q})^2\right]^T \kappa \subset P^T\,\kappa$

In fig. (II-1.3) we have denoted the fermion flow by a gray arrow; in defining a fermion flow separate from fermion-number flow, we follow the formalism of [38] for the calculation of fermion-number-violating interactions. Note that in fermion-number-conserving interactions such as the one in fig. (II-1.2), we can choose fermion flow to equal fermion-number flow, which indicated by the arrows on the fermion lines themselves. Thus, we can drop the separate arrow for such diagrams. Note that there will be instances where we wish to emphasize differences between fermion-number-conserving and -violating interactions, and will keep the gray arrows in such cases. By making use of this fermion flow, we can fix an ordering of vertices and propagators, which also helps us in understanding the structure of the RGE (II-1.8). Just as with fermion-number flow in diagrams that conserve fermion-number, we fix a fermion flow, and then go through the diagram *against* its direction. Thus, renormalization of the left leg of fig. (II-1.3a) leads to the structure $\sim \kappa\,(\tilde{Q})^2 \subset \kappa\,P$, and of the right leg of fig. (II-1.3b), to $\sim \left[(\tilde{Q})^2\right]^T \kappa \subset P^T\,\kappa$. In other words, by having the Z' loop on either leg to the left and right of the Weinberg-Operator as in fig. (II-1.3), we already know that the corresponding contribution to the Weinberg-Operator's renormalization will have the form

$$P^T\,\kappa + \kappa\,P\,. \qquad \text{(II-1.11)}$$

On the other hand, there is one other contribution that renormalizes the Weinberg-Operator due to the Z': *the vertex correction.* we will see has that this contribution leads to a drastic change in the $\beta_{\mathcal{K}}$-function. We show the diagram for the vertex correction coming from the Z' in fig. (II-1.4).

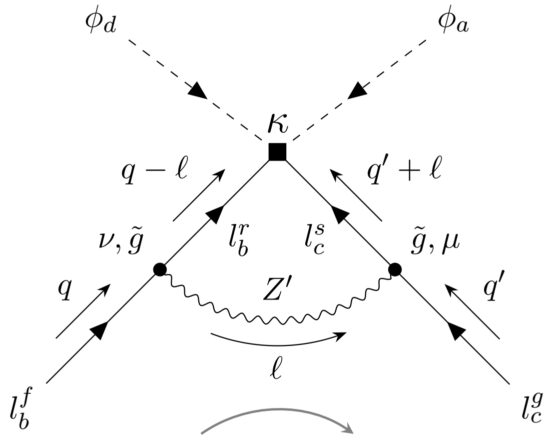
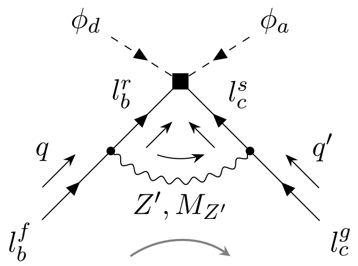


Figure II-1.4.: Z' contribution to the Weinberg-Operator via vertex correction; the superscript is the lepton-family index; the subscript, the $SU(2)_L$ index; the gray arrow denotes fermion flow

We compute this diagram to be given by



$$= (-i\,\tilde{g})^2\,\tilde{Q}_{sg}\,\mathcal{K}_{sr}\,\tilde{Q}_{rf}\cdot E_{abcd}\cdot \mathcal{I}_{\delta\mathcal{K}}(q,q',M_{Z'},d)\,, \qquad \text{(II-1.12)}$$

where $\mathcal{I}_{\delta\mathcal{K}}(q,q',M_{Z'},d)$ is again the d-dimensional loop-integral, and $E_{abcd} = \frac{1}{2}(\varepsilon_{cd}\varepsilon_{ba} + \varepsilon_{ca}\varepsilon_{bd})$ is the $SU(2)_L$ structure of the $\mathcal{K}$-vertex. We thus find for the vertex renormalization constant

$$\delta\mathcal{K}_{,1}\Big|_{U(1)_{L_\mu - L_\tau}} \propto \frac{\tilde{g}^2}{16\pi^2}\left(\tilde{Q}^T\,\mathcal{K}\,\tilde{Q}\right) = \frac{\tilde{g}^2}{16\pi^2}\begin{pmatrix}0 & 0 & 0\\ 0 & \mathcal{K}_{22} & -\mathcal{K}_{23}\\ 0 & -\mathcal{K}_{23} & \mathcal{K}_{23}\end{pmatrix}, \qquad \text{(II-1.13)}$$

which is different from the structure in the Standard Model! And in fact, we can show that **no** matrix P exists such that

$$P^T\,\mathcal{K} + \mathcal{K}\,P \overset{!}{\sim} \tilde{Q}^T\,\mathcal{K}\,\tilde{Q}\,. \qquad \lightning \qquad \text{(II-1.14)}$$

We will generalize this statement further in chapter (II-2). Analogously to before, we can understand how the structure of this term arises from that of the corresponding diagram, which we visualize in fig. (II-1.5). We see that with respect to fermion flow, the Weinberg-Operator is "sandwiched" by Z' gauge-couplings, and thus the respective charge matrix in flavor-space.

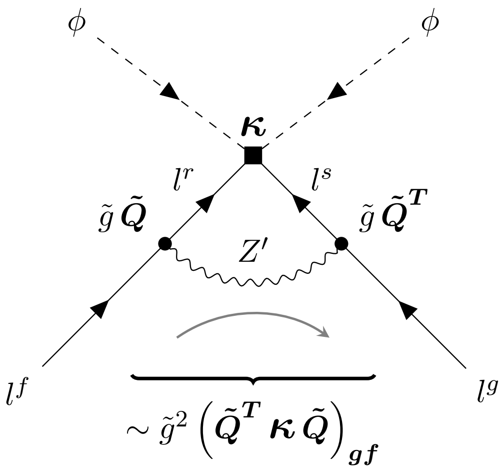


Figure II-1.5.: Structure of the Z' contribution to the Weinberg-Operator via vertex correction; the gray arrow denotes fermion flow

As we see, this interaction structure thus gives us a term of the form $\sim \tilde{Q}^T\,\kappa\,\tilde{Q} \sim G^T\,\kappa\,G$ for a matrix $G \propto \tilde{Q}$, which we can already "read off" from the diagram. Let us elaborate on this new term, what does it mean for the β_κ-function? This new term means that the structure of the β_κ-function is no longer given by eq. (II-1.9), but instead has to be extended to

$$\beta_\kappa = \underbrace{\alpha\,\kappa + P^T\,\kappa + \kappa\,P}_{\text{Standard Model} + U(1)_{L_\mu - L_\tau}} + \underbrace{\boldsymbol{G^T\,\kappa\,G}}_{\boldsymbol{U(1)_{L_\mu - L_\tau}}} \quad ! \tag{II-1.15}$$

Since we also have vertex corrections from gauge bosons in the Standard Model, naturally the following question arises:

Why is there no $\beta_\kappa \supset G^T\,\kappa\,G$ term in the Standard Model β_κ-function?

It turns out that the operator structure corresponding to the new term *is* actually also present in the SM; *however*, the SM gauge interactions are lepton-flavor universal, i.e., they act in the exact same way on all three lepton generations. Therefore, the coupling matrices in lepton-flavor space are proportional to the unit matrix, which trivially commutes with κ. This means that while the structure $G^T\,\kappa\,G$ is in principle present in the SM, it reduces to an $\alpha\,\kappa$ term:

$$G_{SM} \propto \mathbb{1} \implies G_{SM}^T\,\kappa\,G_{SM} \propto \mathbb{1}^T\,\kappa\,\mathbb{1} = \kappa \tag{II-1.16}$$

$$\implies \beta_\kappa \supset G_{SM}^T\,\kappa\,G_{SM} \sim \alpha\,\kappa \quad ! \tag{II-1.17}$$

The critical point in the origin of the new term is that the coupling of the gauge boson is **not** universal for all three lepton-generations. This leads us to the following conclusion, and thus the title of this thesis:

> **Flavor-nonuniversal gauge-extensions** of the Standard Model give rise to **new quantum effects on neutrino parameters** by introducing a **novel term** $\sim \boldsymbol{G^T \kappa G}$ in the $\boldsymbol{\beta_\kappa}$**-function at the one-loop level**. This **extends** the current one-loop **parametrization** of the β_κ-function to
>
> $$\beta_\kappa = \alpha\,\kappa + P^T\,\kappa + \kappa\,P + \boldsymbol{G^T\,\kappa\,G}\,. \tag{II-1.18}$$

Note that while renormalization in, e.g., a $U(1)_{L_\mu - L_\tau}$ gauge extension has been partially considered before, this new effect was not found because the contributions to β_κ coming from the Z' were not considered, and the same structure as in the Standard Model was assumed—see, for instance, [39].

Let us now turn our attention to what this term concretely means for the neutrino parameters. While the detailed derivation and discussion of the changes brought about by the $G^T\,\kappa\,G$ term is left to later chapters, let us state here what its most striking feature is. As discussed in section (I-3.4), we can make use of the leptonic mixing matrix U to diagonalize κ and the β_κ-function. By applying the transformation

$$\kappa \longrightarrow U^*\,\kappa\,U^\dagger = \kappa_{\text{diag}}\,, \tag{II-1.19}$$

we can derive RGEs for the eigenvalues of κ, or in other words, for the neutrino masses. Note again that strictly speaking, the eigenvalues of κ are only proportional to the squared neutrino masses, since we are missing the vacuum expectation value of the Higgs field. We diagonalize the β_κ-function, recalling that $t = \ln \mu/\mu_0$, where μ is the scale we are at, and μ_0 is some reference scale. We thus find the RGEs for the eigenvalues κ_i

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} = \mathrm{Re}\left[\alpha\,\kappa_i + 2\left(U\,P\,U^\dagger\right)_{ii}\kappa_i + \sum_{\boldsymbol{j=1}}^{\boldsymbol{3}}\left[\left(\boldsymbol{U\,G\,U^\dagger}\right)_{\boldsymbol{ji}}\right]^{\boldsymbol{2}}\boldsymbol{\kappa_j}\right] \tag{II-1.20}$$

$$\overset{U\ \text{real}}{=} \alpha\,\kappa_i + 2\left(U\,P\,U^T\right)_{ii}\kappa_i + \sum_{\boldsymbol{j=1}}^{\boldsymbol{3}}\left[\left(\boldsymbol{U\,G\,U^T}\right)_{\boldsymbol{ji}}\right]^{\boldsymbol{2}}\boldsymbol{\kappa_j}\,. \tag{II-1.21}$$

Note that in this equation, we do not sum over repeated indices in the above equation unless explicitly denoted by a summation symbol. Let us take a closer look at what this equation tells us. First, in the SM, we only have the first part; whereas in our gauge-extension, we also have the second:

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} = \mathrm{Re}\Bigg(\underbrace{\alpha\, \boldsymbol{\kappa_i} + 2\left(U\,P\,U^T\right)_{ii} \boldsymbol{\kappa_i}}_{\text{Standard Model} + U(1)_{L_\mu - L_\tau}} + \underbrace{\sum_{j=1}^{3} \left[\left(U\,G\,U^T\right)_{ji}\right]^2 \boldsymbol{\kappa_j}}_{U(1)_{L_\mu - L_\tau}} \Bigg)\,. \qquad \text{(II-1.22)}$$

Second, we see one crucial difference that arises due to the new term. In the SM, all terms on the right-hand side of the equation are proportional to the eigenvalue on the left-hand side; *however*, in our gauge extension, we find a *sum* over *all* eigenvalues on the right-hand side. So, what does this effectively mean for the RGE evolution? This means that on the one hand, in the Standard Model, if an eigenvalue is zero at the scale μ_0, it will *stay* zero throughout the one-loop RGE evolution. On the other hand, in our gauge-extension—thanks to the new $G^T\,\kappa\,G$ term—even if we start with only one non-zero eigenvalue at the scale μ_0, the RGE evolution *generates non-zero eigenvalues at the one-loop level*!

In models with **flavor-nonuniversal gauge-extensions** of the Standard Model, the $\beta_{\boldsymbol{\kappa}}$-function can **raise the rank of the mass matrix at the one-loop level**. This occurs due to the new $G^T\,\kappa\,G$ term, which induces a **sum** over **all** eigenvalues in their β-functions,

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} \;=\; \mathrm{Re}\Bigg(\alpha\,\kappa_i + 2\left(U\,P\,U^\dagger\right)_{ii} \kappa_i + \sum_{\boldsymbol{j=1}}^{\boldsymbol{3}} \left[\left(\boldsymbol{U\,G\,U^\dagger}\right)_{\boldsymbol{ji}}\right]^{\boldsymbol{2}} \boldsymbol{\kappa_j}\Bigg)\,. \qquad \text{(II-1.23)}$$

Raising the rank of the mass matrix corresponds to **generating new non-zero mass eigenvalues.**

Let us visualize the effect we have found here with an explicit example. In fig. (II-1.6) we have plotted the running of the eigenvalues $\kappa_i(t)$ due to the $G^T\,\kappa\,G$ term, starting from a scale $\Lambda = 10^{14}\,\mathrm{GeV}$—corresponding to the right-handed neutrino mass scale—down to the weak scale $\sim 100\,\mathrm{GeV}$. At the scale Λ we started with *only one non-zero eigenvalue*, $\kappa_3(t_\Lambda) = 1$ (in terms of some reference scale), and $\kappa_1(t_\Lambda) = \kappa_2(t_\Lambda) = 0$. To obtain $\kappa(t_\Lambda)$ we set the leptonic mixing matrix $U(t_\Lambda)$ to a random, orthogonal 3×3 matrix, and performed the inverse transformation to eq. (II-1.19), $\kappa(t_\Lambda) = U^T(t_\Lambda)\,\kappa_{\mathrm{diag}}\,U(t_\Lambda)\,.$

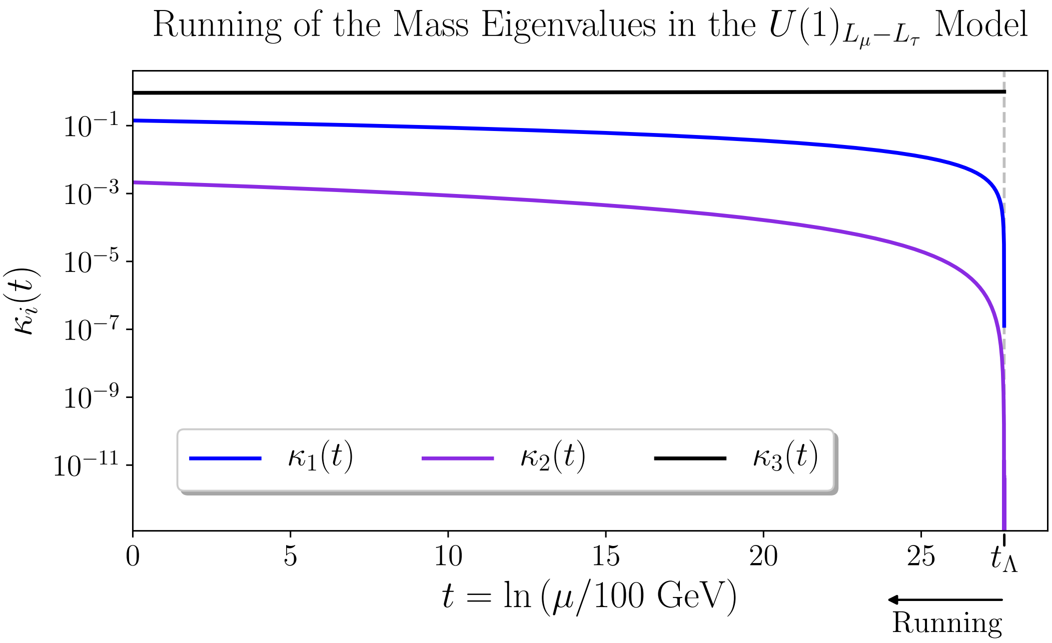


Figure II-1.6.: Running of the eigenvalues κ_i of κ due to the $G^T \kappa G$ term as a function of t; we run the RGEs down from $t_\Lambda = \ln(\Lambda/100\,\text{GeV})$, with $\Lambda = 10^{14}\,\text{GeV}$, and $\tilde{g} = 0.5$; at t_Λ, we set $\kappa_3(t_\Lambda) = 1$, $\kappa_1(t_\Lambda) = \kappa_2(t_\Lambda) = 0$—i.e., only one non-zero eigenvalue in terms of some reference scale—and we set the leptonic mixing matrix $U(t_\Lambda)$ to a random, orthogonal 3×3 matrix

In fig. (II-1.6), we clearly see that even though we started with only *one non-zero eigenvalue* at the scale Λ, we *generate two further eigenvalues* simply by virtue of the RGE running! Here, we see an explicit example of the impact of flavor-nonuniversal gauge interactions on neutrino masses. We furthermore observe that most of the growth happens between $t = t_\Lambda \sim 27.5$ and $t \sim 20$, corresponding to approximately three orders of magnitude from $\mu = \Lambda = 10^{14}\,\text{GeV}$ to $\mu \sim 10^{11}\,\text{GeV}$. This is interesting because the heavy Z' boson needs to be integrated out below its mass scale, and will thus not contribute to the RGEs anymore. This aspect is closely related to the model building and will be discussed further in later chapters. We have nevertheless plotted the running down to the weak scale to elucidate the phenomenon of RGE mass generation, and that most of it occurs during the first few orders of magnitude.

Let us now take a closer look at the mass values we generated through the RGE running—remember that we set $\kappa_3(t_\Lambda) = 1$ meaning the absolute scale is not fixed. We see that because of the RGE running, we obtain eigenvalues $\kappa_3 \sim \mathcal{O}(1)\,,\ \kappa_2 \sim \mathcal{O}(10^{-1})\,,\ \kappa_1 \sim \mathcal{O}(10^{-3})$, which is very intriguing. That is, we observe here that *we may be able to explain the observed mass splittings between the neutrinos due to these relative scales of the initial and generated eigenvalues*! Inspired by the cosmological upper limit on the sum of neutrino masses, $\sum m_\nu < 0.111\,\text{eV}$, let us for instance assume $m_{\nu,3}(t_\Lambda) = 0.08\,\text{eV} = 8 \times 10^{-2}\,\text{eV}$ [6]. We can use the relative size of the generated masses to calculate the mass squared differences, $\Delta m^2_{3\ell} = m^2_{\nu,3} - m^2_{\nu,1\,\text{or}\,2}$ and $\Delta m^2_{21} = m^2_{\nu,2} - m^2_{\nu,1}$. We present the result of this calculation in tab. (II-1.1).

	$m_{\nu,3}$	$m_{\nu,2}$	$m_{\nu,1}$
Mass Values Before Running	$8 \times 10^{-2}\,\text{eV}$	$0\,\text{eV}$	$0\,\text{eV}$
Mass Values After Running	$\sim 8 \times 10^{-2}\,\text{eV}$	$\sim 8 \times 10^{-3}\,\text{eV}$	$\sim 8 \times 10^{-5}\,\text{eV}$
Mass Splittings	$\boldsymbol{\Delta m^2_{3\ell} \sim 6 \times 10^{-3}\,\text{eV}^2}$	$\boldsymbol{\Delta m^2_{21} \sim 6 \times 10^{-5}\,\text{eV}^2}$	
Measured Mass Splittings	$\boldsymbol{\Delta m^2_{3\ell} \sim 7 \times 10^{-3}\,\text{eV}^2}$	$\boldsymbol{\Delta m^2_{21} \sim 3 \times 10^{-5}\,\text{eV}^2}$	

Table II-1.1.: Neutrino mass values before and after RGE running following fig. (II-1.6), and the resulting mass splittings; the one non-zero mass value before running is inspired by the cosmological constraint on the neutrino mass sum [6]; the measured mass splittings are taken from [3, 4]

We see from our analysis in tab. (II-1.1) that by using a physically reasonable mass of 0.08 eV for the one non-zero mass we start with, we obtain an excellent estimate of the neutrino mass splittings we measure today! This means that the *measured mass splittings could very well be explained by an RGE running in a flavor-nonuniversal gauge-extension*! Furthermore, this framework gives a natural way to induce a normal hierarchy, since the two generated mass eigenvalues remain below the original one. Formulated in a more general way, we thus find:

> The **RGE evolution in flavor-nonuniversal gauge-extensions** and emerging generation of eigenvalues of κ provides a framework to **explain and predict the mass splittings $\boldsymbol{\Delta m^2_{3\ell}}$ and $\boldsymbol{\Delta m^2_{21}}$**. Furthermore, generation with just one non-zero mass-eigenvalue at high energy scales may also **naturally induce a normal mass hierarchy** of the neutrinos.

While we did not discuss this aspect in this chapter, note that just as the new quantum effects do not only influence the eigenvalues' running as seen in eq. (II-1.23) There are similar effects on the mixing angles; these effects may, for instance, drive the RGEs for the mixing angles faster, and to different fixed points.

To summarize our results thus far, we list the most crucial findings discussed in this chapter in tab. (II-1.2), and compare our extended model with the SM.

	Standard Model	$\boldsymbol{U(1)_{L_\mu - L_\tau}}$ **Gauge-Extension**
Interaction matrices in flavor-space	$Q_{SM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}$	$\tilde{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \neq \mathbb{1}$
One-loop β_κ structure	$\alpha\,\kappa + P^T\,\kappa + \kappa\,P$	$\alpha\,\kappa + P^T\,\kappa + \kappa\,P + \boldsymbol{G^T\,\kappa\,G}$, with $G \propto \tilde{Q}$
One-loop $d\kappa_i/dt$ structure	$\propto \kappa_i$	$\left(\propto \kappa_i\right) + \boldsymbol{\sum_{j=1}^{3}}\left(\boldsymbol{\propto \kappa_j}\right)$
Rank of κ throughout RGE evolution	constant	increases $\longrightarrow 3$
Mass splittings and mass hierarchy	Ad-hoc	Provides framework to explain and predict them

Table II-1.2.: Summary of core findings distinguishing the RGE evolution and structure of the Weinberg-Operator in the Standard Model and the $U(1)_{L_\mu - L_\tau}$ gauge-extension

Thus far, we have mainly focused on a $U(1)_{L_\mu - L_\tau}$ gauge-extension; however, this is not the only possible flavor-nonuniversal gauge-extension we can consider. Therefore, building on our findings so far, we will generalize our discussion in the next chapter.

CHAPTER

II-2

General Results

Previously, we have seen how the β-function of the Weinberg-Operator is extended in flavor-nonuniversal gauge-extensions of the Standard Model. We have observed that through a novel term of the form $\beta_{\mathcal{K}} \supset G^T\,\kappa\,G$, we can increase the rank of the neutrino mass matrix at the one-loop level. Until now, we considered these phenomena in the framework of a $U(1)_{L_\mu - L_\tau}$ gauge extension, which we will generalize to other flavor-nonuniversal gauge symmetries in this chapter, and elaborate on previously made, general points in more detail.

II-2.1 The Most General Structure of the $\beta_{\mathcal{K}}$-Function: Possible Terms, Their Origin, and Transformation Behavior

II-2.1.1 The Most General $\beta_{\mathcal{K}}$-Function

First, let us consider, from a symmetry standpoint, what the most general possible structure for the RGEs of the Weinberg-Operator at the one-loop level is—i.e., the structure of the one-loop $\beta_{\mathcal{K}}$-function. Let us first state the requirements the RGEs need to fulfill, and explain their origin afterward:

1. The $\beta_{\mathcal{K}}$-function contains only κ, not $\kappa^\dagger$
2. The $\beta_{\mathcal{K}}$-function contains at most one power of κ
3. The $\beta_{\mathcal{K}}$-function is symmetric in flavor-space
4. The $\beta_{\mathcal{K}}$-function transforms in the same way as κ in flavor-space

Let us now explain each of these requirements:

1. : We are looking for contributions to the β_{κ}-function at the one-loop level, this means that we need diagrams that renormalize the interaction shown in fig. (II-1.1) and (II-2.1a). This means that the external (number-) flows of the Higgs-, and lepton-doublet fields have to be *ingoing*. On the other hand, the coupling $\kappa^\dagger$ appears with all external (number-) flows *outgoing*; we show this diagram and the comparison to κ in fig. (II-2.1b). This does not entirely dismiss, for instance, lepton-number-violating interactions—these may appear in self-energy corrections. However, it does require us to go to at least two-loops to internally flip all the flows such that we can use $\kappa^\dagger$. That is because we cannot couple to *all four* legs with some flow-changing interaction at the one-loop level.

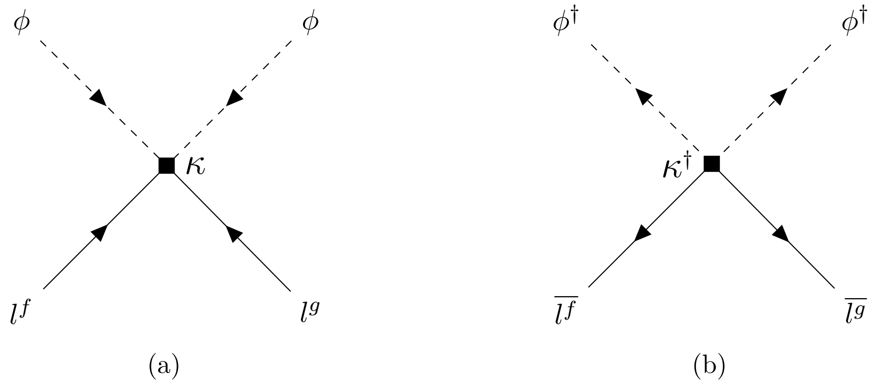


Figure II-2.1.: Comparison of the external (number) flow of the Higgs and lepton fields for κ, fig. (II-2.1a), and $\kappa^\dagger$, fig. (II-2.1b)

Note that if we are in a theory that *does not have any interactions that reverse the Higgs- and lepton-doublets' flow, the form of the β_{κ}-function we derive at the one-loop level here will also hold at the two- and higher-loop level.* This is because, as we will see, requirements 3 and 4 need to be valid to any order in the loop-expansion, and requirement number 2 is in principle also independent of the loop-order.

2. : While this is not a general requirement at the one-loop level in itself—as stated in the previous paragraph—it does stem from an order-expansion. Namely, we are only considering renormalization at the order $1/\Lambda$, where Λ is the UV-scale up to which this effective operator provides a good description of the interaction; for instance, this may be the mass-scale of right-handed neutrinos in a Type-1 Seesaw Mechanism. That is, we do not consider, e.g., double insertions of κ, which would correspond to $\mathcal{O}(1/\Lambda^2)$, since this would require us to also account for higher-dimensional operators—in this example of dimension six. Since these contributions are heavily suppressed, due to the additional power of the high scale Λ, we do not consider them here.

3. : To explain this point, let us recall the form of the Weinberg-Operator,

$$\mathcal{L}_{int} \supset \frac{1}{4}\,\kappa_{gf}\,\overline{l_c^{g,C}}\,\varepsilon^{cd}\,\phi_d\,l_b^f\,\varepsilon^{ba}\,\phi_a\,+\text{h.c.} \sim \kappa_{gf}\,l^g\,l^f\,\phi\,\phi\,+\text{h.c.}\,, \tag{II-2.1}$$

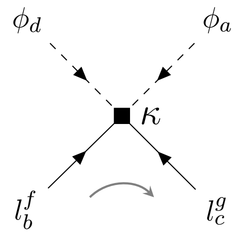
where we have simplified the expression and omitted $SU(2)_L$-indices in the last part. We see that κ is necessarily symmetric, since any antisymmetric component drops out:

$$\kappa_{gf}\, l^g\, l^f \overset{!}{=} -\kappa_{fg}\, l^g\, l^f = -\kappa_{gf}\, l^f\, l^g = -\kappa_{gf}\, l^g\, l^f \tag{II-2.2}$$

$$= 0\,. \tag{II-2.3}$$

This means that any corrections to κ need to be symmetric, too. If we, e.g., write $\kappa(t_2) = \kappa(t_1) + \Delta\kappa(t_1, t_2)$, any antisymmetric component of $\Delta\kappa(t_1, t_2)$ cannot contribute to $\kappa(t_2)$. And since the β_κ-function encodes of the scale evolution of κ, it is required to be symmetric in flavor-space as well. To put it another way, κ is symmetric before the RGE evolution, and should still be symmetric afterward. Hence, since it gives rise to this evolution, the β_κ-function also needs to be symmetric.

In fact, the Weinberg-Operator is symmetric not only in flavor-space, but otherwise as well—both for the lepton, as well as the Higgs fields. Therefore, κ and the β_κ-function are symmetric in *any* internal space. We can, for instance, verify this in $SU(2)_L$-space. While the $SU(2)_L$ coupling structure is defined separately from κ itself, we see transposition symmetry represented in the Feynman-rule, since it is symmetric in the $SU(2)_L$ indices of both the lepton- and Higgs-doublet pairs. Note that we say "transposition symmetry" to make the distinction clear between the internal symmetry itself—e.g., $SU(2)_L$—and the symmetry under transpositions in this space—e.g., the exchange of $SU(2)_L$ indices of the lepton-doublet pair. The exact Feynman-rule is listed in the appendix, but structurally, we have that



$$\sim \quad \kappa_{gf}\, \underbrace{\frac{1}{2}\left(\varepsilon_{cd}\,\varepsilon_{ba} + \varepsilon_{ca}\,\varepsilon_{bd}\right)}_{\substack{\text{symmetric in}\, b \leftrightarrow c \\ \text{and}\, a \leftrightarrow d}}\, P_L\,. \tag{II-2.4}$$

We see from eq. (II-2.4) that the Feynman-rule is indeed symmetric under the exchange of the $SU(2)_L$ indices $b \leftrightarrow c$ of the lepton-doublets, and $a \leftrightarrow d$ of the Higgs-doublets. This is one instance of transposition symmetry in internal spaces besides flavor-space. Similarly, if additional internal spaces are present in some theory, the vertex and β_κ-function are required to be symmetric in these spaces too.

4. : Following the same line of arguments as for requirement number 3, κ follows the same transformation behavior,

$$\kappa(t) \longrightarrow U^*(t)\, \kappa(t)\, U^\dagger(t)\,, \tag{II-2.5}$$

for any value of t, i.e., at any scale. Like before, this means that any corrections due to scale evolution are also required to transform in this way as well, and thus the β_κ-function needs to as well. Note that the structure of the transformation is required to stay the same, the specific values of the entries of U may change—indicated by the t-dependence of U.

We can now use requirements $1-4$ to write down the most general structure the β_κ-function can have at the one-loop level to order $1/\Lambda$. Let us start by writing down the most general form following requirements 1 and 2:

$$\beta_\kappa = \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \alpha\,\kappa + B\,\kappa + \kappa\,C + D_1\,\kappa\,E_1 + D_2\,\kappa\,E_2 + D_3\,\kappa\,E_3 + F\,, \qquad \text{(II-2.6)}$$

where α is a scalar in flavor-space and B, C, $D_{1,2,3}$, $E_{1,2,3}$, and F are matrices. Note that it is enough to write just one each of B and C. Since they are the only matrix factors of κ in their respective terms, and matrix multiplication is associative, we can always write $B_1\,\kappa + B_2\,\kappa \equiv B\,\kappa$, and similarly for C. Therefore, we only need one such term in β_κ. We cannot, however, do the same for the $D_i\,\kappa\,E_i$ terms; nevertheless, for illustrative purposes, we restrict ourselves to just three for now. The F-matrix is κ-independent and would thus, somehow, constitute a κ-independent correction to the Weinberg-Operator at the one-loop level. This correction would need to come from the vertex correction because wave function renormalization leads to B- and C-type terms, as we have seen in the previous chapter. Since such a vertex correction diagram in general has a non-vanishing finite part, this implies, however, the existence of a one-loop realization of the Weinberg-Operator. But such a radiative realization would be independent of the UV-scale Λ, and instead depend on the masses of the internal particles, which are by definition smaller than Λ. Such a term cannot exist since it contradicts the definition of the Weinberg-Operator as the effective operator stemming from integrating out heavy fields below their mass scale Λ. Or in other words, if such a realization existed, it would dominate over κ, since it is less suppressed. This means we could use this radiative mechanism to generate masses, and didn't need the effective description provided by κ above the mass scale of the intermediate particles—at least to the leading order. Therefore, this F-matrix term can be dropped. Note also that this argument actually makes requirement number 2 more stringent by requiring β_κ not only have at most one power of κ, but *exactly* one power of it.

The next step is to make use of requirement number 3, i.e., to enforce symmetry in flavor-space. We thus obtain,

$$\beta_\kappa^T \;=\; \alpha\,\kappa \;+\; \overbrace{\kappa\,B^T}^{\textcircled{1}} \;+\; \overbrace{C^T\,\kappa}^{\textcircled{2}} \;+\; \overbrace{E_1^T\,\kappa\,D_1^T}^{\textcircled{3}} \;+\; \overbrace{E_2^T\,\kappa\,D_2^T}^{\textcircled{4}} \;+\; \overbrace{E_3^T\,\kappa\,D_3^T}^{\textcircled{5}} \qquad \text{(II-2.7)}$$

$$\overset{!}{=}\; \alpha\,\kappa \;+\; \underbrace{B\,\kappa}_{\textcircled{2}} \;+\; \underbrace{\kappa\,C}_{\textcircled{1}} \;+\; \underbrace{D_1\,\kappa\,E_1}_{\textcircled{3}} \;+\; \underbrace{D_2\,\kappa\,E_2}_{\textcircled{5}} \;+\; \underbrace{D_3\,\kappa\,E_3}_{\textcircled{4}} \;=\; \beta_\kappa \qquad \text{(II-2.8)}$$

We can now compare eq. (II-2.7) and (II-2.8) to obtain conditions on the B, C, D_i, and E_j matrices. At this point, we will consider the second and third $D_i\,\kappa\,E_i$ as one connected unit to obtain the most general structure, which is the reason behind choosing to start with exactly three such terms. By comparing the terms marked with the same circled numbers in eq. (II-2.7) and (II-2.8), we obtain the following conditions:

- ① and ②: $B = C^T$
- ③ : $D_1 = E_1^T$
- ④ and ⑤: $D_2 = E_3^T$ and $D_3 = E_2^T$,

which leads us to the form

$$\beta_{\mathcal{K}} = \alpha\,\kappa + C^T\,\kappa + \kappa\,C + E_1^T\,\kappa\,E_1 + E_2^T\,\kappa\,E_3 + E_3^T\,\kappa\,E_2\,. \tag{II-2.9}$$

We see that we obtain the $P^T\,\kappa + \kappa\,P$ structure we have seen in the previous chapter from the originally B- and C-type terms. The first $D_i\,\kappa\,E_i$ term is itself symmetric and corresponds to the $G^T\,\kappa\,G$ term. What we also observe is that in addition to these two matrix terms and the scalar term $\alpha\,\kappa$, the $D_{2,3}\,\kappa\,E_{2,3}$ terms lead to a new, coupled term of the form $G_+^T\,\kappa\,G_- + G_-^T\,\kappa\,G_+$, which is only symmetric when adding *both* sub-terms. For consistency with the final equation we will present at the end and the SM discussion in the previous part, let us rename the matrices. Eq. (II-2.9) thus becomes

$$\beta_{\mathcal{K}} = \alpha\,\kappa + P^T\,\kappa + \kappa\,P + G^T\,\kappa\,G + \frac{1}{2}\left(G_+^T\,\kappa\,G_- + G_-^T\,\kappa\,G_+\right). \tag{II-2.10}$$

The reason for the indexation with $\pm$ and the normalization factor of $1/2$ will become apparent when addressing the origin of the various terms in $\beta_{\mathcal{K}}$, soon. While we did not go into detail about this here because we are mostly interested in the flavor structure, α is required to be symmetric in other internal spaces besides flavor-space if they are present—the other matrices in their respective combinations as well. This follows from the discussion at the end of the one for requirement number 3.

Lastly, we use requirement number 4 to analyze the form and transformation behavior of the P, G, and $G_\pm$ matrices. To that end, let us perform the transformation $\kappa \longrightarrow U^*\,\kappa\,U^\dagger$, and enforce the correct overall transformation behavior of the $\beta_{\mathcal{K}}$-function, $\beta_{\mathcal{K}} \longrightarrow U^*\,\beta_{\mathcal{K}} U^\dagger$. We denote the transformed P, G, and $G_\pm$ matrices with a tilde. Since the different parts of $\beta_{\mathcal{K}}$ do not mix under the flavor-space transformation, we can consider them independently—apart from the sub-terms related via transposition. Applying the transformations thus yields:

$$\alpha\, \kappa \longrightarrow \tilde{\alpha}\, U^*\, \kappa\, U^\dagger \stackrel{!}{=} U^*\, \alpha\, \kappa\, U^\dagger\,, \tag{II-2.11}$$

$$\begin{aligned} P^T\, \kappa + \kappa\, P \longrightarrow & \overset{\downarrow}{\tilde{P}}{}^T\, U^*\, \kappa\, U^\dagger + U^*\, \kappa\, U^\dagger\, \overset{\downarrow}{\tilde{P}} \\ & \stackrel{!}{=} U^*\, P^T\, \kappa\, U^\dagger + U^*\, \kappa\, P\, U^\dagger\,, \end{aligned} \tag{II-2.12}$$

$$G^T\, \kappa\, G \longrightarrow \overset{\downarrow}{\tilde{G}}{}^T U^*\, \kappa\, U^\dagger\, \overset{\downarrow}{\tilde{G}} \stackrel{!}{=} U^*\, G^T\, \kappa\, G\, U^\dagger\,, \tag{II-2.13}$$

$$\begin{aligned} G_+^T\, \kappa\, G_- + G_-^T\, \kappa\, G_+ \longrightarrow & \overset{\downarrow}{\tilde{G}}{}_+^T\, U^*\, \kappa\, U^\dagger\, \overset{\downarrow}{\tilde{G}_-} + \overset{\downarrow}{\tilde{G}}{}_-^T\, U^*\, \kappa\, U^\dagger\, \overset{\downarrow}{\tilde{G}_+} \\ & \stackrel{!}{=} U^*\, G_+^T\, \kappa\, G_-\, U^\dagger + U^*\, G_-^T\, \kappa\, G_+\, U^\dagger\,. \end{aligned} \tag{II-2.14}$$

Eq. (II-2.11) is trivially satisfied, since α is by definition a scalar in flavor-space. By inserting identities in the form $\mathbb{1} = U\, U^\dagger = U^*\, U^T$ at the arrows above, and comparing the primed to the unprimed matrices, we obtain from eq. (II-2.11)–(II-2.14):

$$\alpha \longrightarrow\ \alpha\,, \tag{II-2.15}$$

$$P \longrightarrow\ U\, P\, U^\dagger\,, \tag{II-2.16}$$

$$G \longrightarrow\ U\, G\, U^\dagger\,, \tag{II-2.17}$$

$$G_\pm \longrightarrow\ U\, G_\pm\, U^\dagger\,. \tag{II-2.18}$$

This transformation behavior means that the interactions giving rise to the matrices P, G, and $G_\pm$ are overall lepton-number conserving because they transform with one U and one $U^\dagger$. To clarify this point, consider the transformation of a lepton-number-conserving interaction, and a lepton-number-violating one:

$$\begin{aligned} \text{Lepton-number conserving:} \qquad \bar{l}\, A\, l \longrightarrow & \; \bar{l}\, U^\dagger\, \tilde{A}\, U\, l \stackrel{!}{=} \bar{l}\, A\, l \\ \Longrightarrow & \; \tilde{A} = U\, A\, U^\dagger\,, \end{aligned} \tag{II-2.19}$$

$$\begin{aligned} \text{Lepton-number violating:} \qquad l^T\, A\, l \longrightarrow & \; l^T\, U^T\, \tilde{A}\, U\, l \stackrel{!}{=} l^T\, A\, l \\ \Longrightarrow & \; \tilde{A} = U^*\, A\, U^\dagger\,. \end{aligned} \tag{II-2.20}$$

We see that the transformation behavior with one U and one $U^\dagger$—or equivalently one U^* and one U^T—corresponds to a *lepton-number-conserving* interaction, while a combination of U^* and $U^\dagger$—or U and U^T—corresponds to a *lepton-number-violating* interaction.

After having discussed the possible terms in the $\beta_{\mathcal{K}}$-function, restricted by symmetry and other fundamental requirements, as well as these terms' overall transformation behavior, we are now ready to write down our final result for $\beta_{\mathcal{K}}$. Since we did not at any point assume the dimension of κ in flavor-space, this formula is valid for *any* number of lepton generations. We generalize $\sim G^T\, \kappa\, G$ and $\sim G_{\pm}^T\, \kappa\, G_{\mp}$ to an arbitrary number of such terms, denoted by the bracketed superscripts (r) and (s). The previously discussed properties apply to all of them individually. Note that we claim properties concerning hermiticity, etc. of the appearing constituents, which we have not yet shown. Furthermore, we mention which types of diagrams can, in general, contribute—and which cannot. At this point, we already anticipate these properties to display the most salient aspects comprehensively; we will, however, prove this in detail soon.

Note that we could, in principle, add a factor of ± 1 to the $\sim G^{(r)}$ and $\sim G_{\pm}^{(s)}$ terms. This would allow us to keep, e.g., the hermiticity property in full generality by permitting overall minus signs in front of the contributions. Alternatively, we could also allow for anti-hermiticity—i.e., $i\cdot$hermitian—and absorb the overall minus sign in the form of a factor of i into the definition of the matrices themselves. However, we will show by explicit calculation that the sign in front of these terms is, in fact, positive. Note that one may intuitively expect this to depend on the choice of coupling matrices of the interactions giving rise to the $\sim G_{(\pm)}$ terms. For instance, we could choose anti-hermitian couplings, which would seemingly lead to a minus sign; however, such a choice for the generators would then also necessarily change the Feynman-rules such that the effect would cancel out overall. This can also be understood as a consequence of reparametrization invariance—i.e., invariance of the theory under reparametrizations of the fields—meaning that the physical $\beta_{\mathcal{K}}$-function cannot depend on our choice of parametrization for the intermediate particles and their couplings. Note that this concerns the *parametrization* of the interactions, not which interactions are fundamentally present. We will comment more on reparametrization invariance and its effects in this context in subsequent discussions.

Thus, we ultimately obtain:

The **most general, symmetry-allowed, $\boldsymbol{\beta_{\kappa}}$-function for any number of lepton generations, at the one-loop level, and at order $\boldsymbol{1/\Lambda}$** is given by

$$\beta_{\kappa} = \alpha\,\kappa + P^T\,\kappa + \kappa\,P + \sum_r \left(G^{(r)}\right)^T \kappa\,G^{(r)} + \\ + \frac{1}{2}\sum_s \left[\left(G_+^{(s)}\right)^T \kappa\,G_-^{(s)} + \left(G_-^{(s)}\right)^T \kappa\,G_+^{(s)}\right]. \tag{II-2.21}$$

$\boldsymbol{\alpha}$ is a **scalar in flavor-space**; $\boldsymbol{P}$, $\boldsymbol{G^{(r)}}$, and $\boldsymbol{G_{\pm}^{(s)}}$ are **matrices** in flavor-space.

At the one-loop level, only bubble and triangle diagrams can contribute to $\boldsymbol{\beta_{\kappa}}$. They enter via wave function renormalization and vertex corrections, repsectively. The quartic Higgs self-coupling bubble is the only non-triangle that can contribute to vertex corrections. Tadpole, box and higher-point diagrams cannot contribute to the renormalization of $\boldsymbol{\kappa}$.

In the absence of Higgs- or lepton-number-flow-changing interactions eq. (II-2.21) holds also at the two- and higher-loop level at order $\boldsymbol{1/\Lambda}$.

The constituents $\boldsymbol{\alpha}$**,** $\boldsymbol{P}$**,** $\boldsymbol{G^{(r)}}$, and $\boldsymbol{G_{\pm}^{(s)}}$ appearing in the β_{κ}-function **satisfy** the following **properties** and **transformation behaviors**. The **properties of eq. (II-2.22)–(II-2.25)** and the **transposition symmetry of the various terms in $\boldsymbol{\beta_{\kappa}}$ also hold in any additional internal space besides flavor-space**.

$$\alpha^* = \alpha = \alpha^T = \alpha^\dagger\,, \tag{II-2.22}$$

$$P^\dagger = P\,, \tag{II-2.23}$$

$$\left(G^{(r)}\right)^\dagger = G^{(r)}\,, \tag{II-2.24}$$

$$\left(G_{\pm}^{(s)}\right)^\dagger = G_{\mp}^{(s)}\,; \tag{II-2.25}$$

$$\alpha \longrightarrow \alpha\,, \tag{II-2.26}$$

$$P \longrightarrow U\,P\,U^\dagger\,, \tag{II-2.27}$$

$$G^{(r)} \longrightarrow U\,G^{(r)}\,U^\dagger\,, \quad \text{and} \tag{II-2.28}$$

$$G_{\pm}^{(s)} \longrightarrow U\,G_{\pm}^{(s)}\,U^\dagger\,. \tag{II-2.29}$$

II-2.1.2 Origin and Decomposition of the $\beta_{\mathcal{K}}$ Constituents

Let us now investigate the specific form and origin of each of α and the P, G, and $G_\pm$ matrices. To that end, we need to consider the maximum number of vertices that can lead to the respective matrices in $\beta_{\mathcal{K}}$. We can do this via general loop-topological considerations, as well as our knowledge of renormalization.

In principle, since we are investigating corrections to the Weinberg-Operator, which has four external legs, we could have at most box diagrams at the one-loop level. Pentagon or higher-point diagrams cannot contribute via additional external particles, as this would constitute a different operator. Therefore, this would require us to close additional legs to loops, which would thus be at least of the two-loop order, overall. Furthermore, since κ is contained in all terms of $\beta_{\mathcal{K}}$, at most two external legs can be connected via a loop. If we connected three or four of the external legs, the additional κ-coupling with four fields would yet again lead to—at least—an overall two-loop diagram. This means that we can only have either tadpole-, bubble-, or triangle-diagrams. Therefore, we can have *at most two additional interaction vertices involving three or more fields*, apart from the one given by κ. That is because at least two legs are not participating in any one-loop diagram, and are only involved via the Weinberg-Operator.

Thus, we are left with at most either one vertex on each of the two remaining legs, one leg with two vertices, or a quartic self-coupling involving both of the external Higgs fields. We exclude tadpole diagrams on the external legs, as these do not directly contribute to the renormalization of κ. They are momentum-independent and thus only lead to mass-renormalization of the respective external particle, i.e., not its wave function renormalization, or vertex renormalization of κ. This leaves us with *just bubble diagrams for wave function renormalization, and triangle diagrams and the quartic Higgs self-coupling bubble for vertex corrections.* Note also that at this point, we are only considering interactions of at least three fields; we will discuss two-point insertions when discussing the realness, hermiticity, and conjugacy relations of the $\beta_{\mathcal{K}}$ constituents. Let us now make use of this knowledge to investigate the structure and nature of α, and the P, G, and $G_\pm$ matrices.

***P* :** The P matrix appears as only matrix factor of κ in $P^T\,\kappa + \kappa\,P$, and can thus contain up to two sub-matrices coming from two separate interaction vertices:

$$P = A + B\,C\,, \tag{II-2.30}$$

with

$$P \longrightarrow \tilde{P} = \tilde{A} + \tilde{B}\,\tilde{C} \stackrel{!}{=} U\,A\,U^\dagger + U\,B\,C\,U^\dagger\,. \tag{II-2.31}$$

Note that since A appears as a single matrix, we may add all various contributions we may get of this form into the definition of A. However, since $B\,C$ is a product of two matrices, we cannot in general use this one term to sum up all contributions of this structure. Nevertheless, for simplicity, we will only write one term representatively for all such terms when discussing P, since the properties we find apply to every term in the same way.

The A term can in principle originate from one matrix-valued and one scalar interaction in flavor-space—consider for instance a particle coupling to one Higgs doublet, which is a flavor-singlet, and flavor-nonuniversally to lepton-doublets. The transformation behavior then requires the interaction giving rise to A be lepton-number conserving,

$$A \longrightarrow U\,A\,U^\dagger\,. \tag{II-2.32}$$

On the other hand, the $B\,C$ term arises from two matrix-valued interactions, where both need to be either lepton-number conserving or violating, and may transform with two different flavor matrices:

$$B\,C \longrightarrow U\,B\,\tilde{U}^\dagger\,\tilde{U}\,C\,U^\dagger \quad \text{or} \tag{II-2.33}$$

$$\longrightarrow U\,B\,\tilde{U}^*\,\tilde{U}^T\,C\,U^\dagger\,, \tag{II-2.34}$$

where the unitary matrix $\tilde{U}$ may or may not fulfill $\tilde{U} = U$ or $\tilde{U} = \mathbb{1}$. In the case $\tilde{U} = U$, B and C arise from interactions coupling to two lepton-doublets; and in the case of $\tilde{U} = \mathbb{1}$, they arise from interactions coupling to only one lepton-doublet and no other flavor-dependent field. Furthermore, in fact, eq. (II-2.33) with $\tilde{U} \neq U$ appears in the Standard Model! Namely, it arises from vertex correction due to two Yukawa-coupling insertions, which we have seen in section (I-3.4). In this contribution, the matrices are given by the leptonic Yukawa-coupling matrices, $B = Y_e^\dagger$ and $C = Y_e$. In this case, $\tilde{U} = U_{e_R}$ is the flavor-transformation matrix of the right-handed charged leptons.

We may also imagine a lepton-number-violating interaction for P, by considering a bubble diagram contributing to the wave function renormalization, as drawn it fig. (II-2.2). Note that the intermediate fermion needs to be left-handed for Lorentz-invariance of the interaction term, $\sim \overline{f_L^C}\,l\,S$, where f_L is some left-handed fermion and S some scalar.

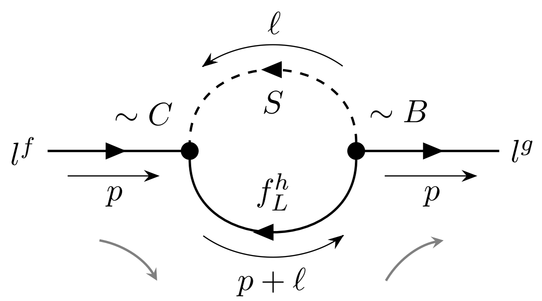


Figure II-2.2.: Wave function renormalization diagram for the lepton doublet due to some scalar S and fermion f_L^h with lepton-number violating interactions; f_L^h is left-handed and carries lepton-flavor h; while B and C are technically not the coupling matrices, they are denoted as such here for illustrative purposes; ℓ is the d-dimensional loop momentum; the gray arrows denote fermion flow

An example of such an interaction is, for instance, taking S from fig. (II-2.2) to be a scalar $SU(2)_L$-triplet, and f^h to be a lepton-doublet.

Note that, in fact, we may even partly simplify, and generalize eq. (II-2.30) further using QFT and renormalization arguments. However, we will do this collectively after also discussing the origin of α, G, and $G_\pm$. We will summarize the one-loop topologies that can give rise to these contributions in terms of Feynman-diagrams, and use them to visualize the arguments used to generalize these constituent decompositions, as well as prove the anticipated hermiticity etc.

$\boldsymbol{\alpha}$: Since $\tilde{\alpha} = \alpha$—i.e., α is by definition a scalar in flavor-space—the interactions giving rise to α need to either not transform in flavor-space, or their transformation behavior must cancel over all. To see how this structure could arise, we can consider, for instance, the flavor-space determinant and trace of α. Since it is a scalar, both of these quantities are simply given by α itself; however, this can give us a hint as to the possible structures α can be made up of. Namely, the trace can be used as an inner (or scalar-) product in flavor-space—for instance, $(A\,,\,B) \equiv \mathrm{Tr}(A\,B^\dagger)$ satisfies linearity, conjugate symmetry, and positive definiteness, as one can verify straightforwardly. This means that since the trace gives us an inner product, taking the trace of a unitarily transforming matrix in flavor-space yields a flavor scalar, $\mathrm{Tr}(A) \longrightarrow \mathrm{Tr}(U\,A\,U^\dagger) = \mathrm{Tr}(U^\dagger\,U\,A) = \mathrm{Tr}(A)$, due to the cyclicity of the trace. This is important because it is essentially the analogous property to a vector in three-dimensional space having the same length independently of the chosen coordinate system. Note that while the determinant is also invariant under flavor transformations due to $\det(U^\dagger) = \det^{-1}(U)$, it cannot be used as an inner product since it is not, for instance, linear: $\det(A\,B) \neq \det(A) + \det(B)$. Following this line of argument, we can decompose α as follows:

$$\alpha = \varphi + \mathrm{Tr}\left(\tilde{A}\right) + \mathrm{Tr}\left(\tilde{B}\,\tilde{C}\right) \tag{II-2.35}$$

where φ is a scalar quantity, and $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are matrices in flavor-space. Note that for the discussion of α here, and when covering the hermiticity etc. later on, we will use the same letters as for matrices in P but with a tilde. We do this to emphasize the similar structure, while keeping the matrices general and distinct. In particular, the tilde does not refer to transformed quantities in this discussion.

As for P, we should in principle add more terms of the form $\mathrm{Tr}\left(\tilde{B}\,\tilde{C}\right)$, but will not do this here for simplicity because all discussed properties hold for every term in the same way. φ, since it is a scalar quantity, arises by definition from interactions that are entirely flavor-independent. This can, for instance, be gauge-boson or quark-loop contributions to the Higgs field-renormalization constant; vertex corrections to κ from, e.g., flavor-universal gauge-bosons or the quartic Higgs vertex; or flavor-universal fermion interactions, etc. The matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ exhibit similar transformation behavior as A, B, and C from the discussion of P—and as stated above—hence the related naming. However, the diagrams giving rise to the trace terms are necessarily different, given that the trace needs to originate from somewhere. Let us first discuss the transformation behavior, and then the origin of the trace.

Since the traces need to be invariant under flavor transformations, the products of transformation matrices to the left and right of $\tilde{A}$ and $\tilde{B}\,\tilde{C}$ need to give unity via the trace's cyclicity. For $\tilde{A}$ and $\tilde{B}\,\tilde{C}$, this means:

$$\tilde{A} \longrightarrow U\,\tilde{A}\,U^\dagger \quad \text{or, e.g.,} \tag{II-2.36}$$

$$\longrightarrow U^*\,\tilde{A}\,U^T\,, \tag{II-2.37}$$

$$\tilde{B}\,\tilde{C} \longrightarrow U\,\tilde{B}\,\tilde{U}^\dagger\,\tilde{U}\,\tilde{C}\,U^\dagger \tag{II-2.38}$$

$$\longrightarrow U^*\,\tilde{B}\,\tilde{U}\,\tilde{U}^\dagger\,\tilde{C}\,U^T \quad \text{or, e.g.,} \tag{II-2.39}$$

$$\longrightarrow U^*\,\tilde{B}\,\tilde{U}^*\,\tilde{U}^T\,\tilde{C}\,U^T\,, \quad \text{etc.} \tag{II-2.40}$$

In other words, the interactions giving rise to $\tilde{A}$ must be lepton-number conserving, but it is not fixed whether they transform with U and $U^\dagger$, or U^* and U^T. The interactions giving rise to $\tilde{B}$ and $\tilde{C}$ can be—as for P—either both lepton-number conserving or violating, but just like for $\tilde{A}$, it is again not fixed whether they transform with U and $U^\dagger$, or U^* and U^T. And, as with P before, $\tilde{U}$ may or may not be equal to U or $\mathbb{1}$.

Naturally, the question arises where these trace terms can come from. To answer this question, let us consider the meaning of the trace. Since we are in flavor-space, taking the trace means summing over all flavors; this is necessary when considering loops, since summation over all possible intermediate particles is required. If flavor-dependent fields are external to the respective loop diagram, their flavor is fixed, meaning that we do not take a trace over them. Therefore, loops of leptons—or lepton-flavor-carrying fields—are the only possibility to realize this flavor trace, summing over all lepton-flavor-carrying, internal particles. This, however, also implies that

$$\tilde{A} = 0\,, \tag{II-2.41}$$

since we cannot have one-particle-irreducible diagrams contributing to $\beta_\mathcal{K}$ with internally running, lepton-flavor carrying particles where only one vertex is flavor-dependent. If the particles are created in the loop, they must be annihilated as well, such that two flavor-dependent vertices are required—note again that a tadpole diagram cannot contribute to field renormalization due to its momentum-independence. Fig. (II-2.3) visualizes the argument made previously concerning the origin of the trace lying in summing over internally running particles using a one-loop diagram for the Higgs field-renormalization. Comparing this to the Standard Model $\beta_\mathcal{K}$-function we have seen in section (I-3.4), we see that the loop with a lepton-doublet and a right-handed, charged lepton for the Higgs field-renormalization leads to a trace over Yukawa-coupling matrices, $\sim \mathrm{Tr}(Y_e^\dagger\,Y_e)$.

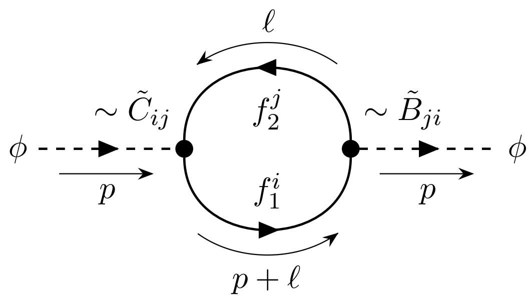


Figure II-2.3.: Wave function renormalization diagram for the Higgs-doublet due to some fermions f_1^i and f_2^j carrying lepton-flavor i and j, respectively; the product of coupling matrices leads to a sum over i and j, and thus a trace—sum over all possible virtual particles needed; while $\tilde{B}$ and $\tilde{C}$ are technically not the coupling matrices, they are denoted as such here for illustrative purposes; ℓ is the d-dimensional loop momentum

Let us also re-state that, following our notation here, the matrix traces refer to lepton-flavor space, and potentially *additional* spaces. Traces over, e.g., *only* quark-flavor space as in the Standard Model, are therefore by definition considered as a part of φ.

Similarly to P, we can further partially simplify, and generalize for α using QFT and renormalization arguments, which we will discuss collectively for all terms soon.

***G* :** We have previously argued that at the one-loop level, we can have at most two additional vertices apart from κ. For the terms $G^T \kappa G$ and $G_+^T \kappa G_- + G_- \kappa G_+$, this means that each of the G and $G_\pm$ *cannot* be broken down into sub-matrices from separate vertices, but originate from just *one interaction each.* Thus, due to eq. (II-2.28) and (II-2.29), this also means that the G, and $G_\pm$ can only originate from *lepton-number-conserving* interactions. We can furthermore show that, in fact, this implies that we can only have vector-interactions! The main components of this proof are that

- the interaction needs to be able to accommodate both an external lepton-doublet and one entering the κ-vertex;
- we need to conserve lepton-number, meaning that two fermions with opposite lepton-number are needed since the G, and $G_\pm$ transform unitarily under U;
- the interaction term needs to be Lorentz-invariant;
- the interaction term needs to be gauge-invariant under the Standard Model.

Since we need at least two fermions to be able to build a Lorentz-invariant interaction term, and require at least three fields for an interaction vertex, the only renormalizable terms we can write down are of mass-dimension four with exactly two fermions, and one scalar or vector field. Therefore, we have the following possible interaction terms:

$$
\begin{gathered}
\sim g_{ij}\,\overline{l_i^C}\,l_j\,S\,,\;\sim g_{ij}\,\overline{f_i^C}\,l_j\,S\,,\;\sim g_{ij}\,\overline{l_i}\,l_j\,S\,,\;\sim g_{ij}\,\overline{f_i}\,l_j\,S\,,\\
\sim g_{ij}\,\overline{l_i^C}\,\gamma^\mu\,l_j\,V_\mu\,,\;\sim g_{ij}\,\overline{f_i^C}\,\gamma^\mu\,l_j\,V_\mu\,,\;\sim g_{ij}\,\overline{l_i}\,\gamma^\mu\,l_j\,V_\mu\,,\;\sim g_{ij}\,\overline{f_i}\,\gamma^\mu\,l_j\,V_\mu\,,
\end{gathered}
\tag{II-2.42}
$$

where f_i is some other (left- or right-handed) fermion, S is some scalar, V_μ some vector field, and we use g representatively for $\sim G$ and $\sim G_\pm$ here. While G and $G_\pm$ are not the coupling matrices, they are directly related to them, hence the "$\sim$" symbol. Note that pseudo-scalar and pseudo-vector interactions $\sim\,\gamma^5$ are also possible, but we treat them only implicitly at this point. Starting from these terms, we can now enforce our above requirements:

- Since we need to accommodate two lepton-doublets, all terms with other fermions are excluded.

  $\Longrightarrow\;\cancel{\sim g_{ij}\,\overline{f_i^C}\,l_j\,S},\;\cancel{\sim g_{ij}\,\overline{f_i}\,l_j\,S},\;\cancel{\sim g_{ij}\,\overline{f_i^C}\,\gamma^\mu\,l_j\,V_\mu},\;\cancel{\sim g_{ij}\,\overline{f_i}\,\gamma^\mu\,l_j\,V_\mu}$
- Since we need to conserve lepton-number, all terms with charge-conjugates are excluded.

  $\Longrightarrow\;\cancel{\sim g_{ij}\,\overline{l_i^C}\,l_j\,S},\;\cancel{\sim g_{ij}\,\overline{l_i^C}\,\gamma^\mu\,l_j\,V_\mu}$
- Since we need to be Lorentz-invariant, all terms with and adjoint and regular fermion of the same chirality are excluded.

  $\Longrightarrow\;\cancel{\sim g_{ij}\,\overline{l_i}\,l_j\,S}$
- Gauge-invariance under the Standard Model does not a priori eliminate any additional term

  $\Longrightarrow\;\sim g_{ij}\,\overline{l_i}\,\gamma^\mu\,l_j\,V_\mu$ ✓

In particular, the exclusion of $\sim\,g_{ij}\,\overline{l_i^C}\,l_j\,S$ means that we *cannot* produce terms such as $\sim G^T\,\kappa\,G$ in the β_κ-function using scalar $SU(2)_L$ triplets. The pictorial reason is that this coupling, while gauge-invariant under the Standard Model and Lorentz-invariant, changes the lepton-number flow, such that we cannot build a coherent vertex-correction diagram using the scalar triplet. Note that while this is applied in particular to the lepton-doublets because we are interested in the flavor structure of β_κ, we cannot use the triplet scalar for a vertex correction with the Higgs either, due to similar reasons concerning flow.

This leaves us with just the *vector-type* interaction term, $\sim g_{ij}\,\overline{l_i}\,\gamma^\mu\,l_j\,V_\mu$. If we also take the pseudo-vector coupling into account, we have two possible interactions,

$$
\mathcal{L}_{int} \sim g_{ij}\,\overline{l_i}\,\gamma^\mu l_j\,V_\mu + g^5_{ij}\,\overline{l_i}\,\gamma^\mu\,\gamma^5 l_j\,V_\mu\,,
\tag{II-2.43}
$$

where g^5_{ij} is the coupling matrix for the pseudo-vector interaction. Using the left- and right-handed projection operators, $P_{L,R} = (\mathbb{1} \mp \gamma^5)/2$, we can, however, simplify this further:

$$\mathcal{L} \sim \overline{l_i}\, \gamma^\mu \left[g_{ij} + g^5_{ij}\, \gamma^5\right] l_j\, V_\mu \tag{II-2.44}$$

$$= \overline{l_i}\, \gamma^\mu \Big[\underbrace{(g_{ij} - g^5_{ij})}_{\equiv \tilde{g}_{ij}} P_L + (g_{ij} + g^5_{ij})\, \underbrace{P_R \Big] l_j}_{=0} V_\mu \tag{II-2.45}$$

$$= \tilde{g}_{ij}\, \overline{l_i}\, \gamma^\mu\, l_j\, V_\mu \tag{II-2.46}$$

As we can see, since the lepton-doublets are purely left-handed, only a vector interaction remains in the end. Furthermore, this means that, generically, we cannot distinguish between a vector interaction with g_{ij}, or a vector and pseudo-vector interaction with effective vector coupling $\tilde{g}_{ij}$. Since the explicit values of the couplings depend on some initial condition, measurement, etc. for both realizations, we cannot know which case is realized at the level of the general form of $\beta_{\mathcal{K}}$. In the context of an explicit model, however, we build it in a certain way with proposed interactions and couplings, and thus may know where couplings originate from, or how to extract them.

For instance, let us assume we also have the same operator with right-handed (charged) leptons, and that they couple in the same way to the vector field as the left-handed lepton-doublets. In that case, we can determine the right- and left-, effective couplings $(g_{ij} \pm g^5_{ij})$, e.g., via a measurement. From their sum and difference, we can then determine the vector and axial couplings, g_{ij} and g^5_{ij}, respectively. Nevertheless, ultimately, the only interaction we have that can give rise to $\sim G^T\, \kappa\, G$ or $\sim G^T_\pm\, \kappa\, G_\mp$ terms, is a vector coupling structure between the left-handed lepton-doublets and the vector field V_μ .

Let us also note at this point the difference in the origin of the $\sim G^T\, \kappa\, G$ and $\sim G^T_\pm\, \kappa\, G_\mp$ terms. While both originate from a vector interaction, they are evidently somewhat different, given that one depends solely on one matrix, G, while the other combines two matrices, G_+ and G_-, in a coupled sum of two terms. However, therein lies also the key to understanding these terms. Namely, the fact that $\sim G^T\, \kappa\, G$ is fully symmetric on its own can be understood in the way that we may mirror the vertex correction diagram giving rise to it, without changing anything; on the other hand, mirroring the diagram for, e.g., the $\sim G^T_+\, \kappa\, G_-$ term necessarily gives $\sim G^T_-\, \kappa\, G_+$ since their sum needs to be symmetric. This tells us that the diagram for each of these terms on their own has a "direction", so to speak, which is exactly opposite to that of the other term.

In other words, they have a *charge flow*, carried by *charged* vector bosons, while the $\sim G^T\, \kappa\, G$ terms arise from *neutral* vector bosons. In other words, the $G_\pm$ arise from charged vector bosons and their antiparticles coupling to the left-handed lepton-doublets. Note that this charge does *not* refer to electric charge, but to *flavor-charge*. This also gives us a clear view on the meaning of G and $G_\pm$, and in what class of theories they can arise, namely such with *flavor-nonuniversally interacting, flavor-neutral and flavor-charged vector bosons*, respectively. Furthermore, this gives us a hint about the relation between G_+ and G_-; namely, that they arise from coupling to the charged vector boson and its antiparticle. We can understand this again from the symmetry of the diagrams: an incoming charged vector boson at a vertex corresponds to the outgoing antiparticle of said vector boson. This is responsible for the different—but related—coupling at the two vertices.

We can, for instance, choose the positively charged vector boson to be outgoing at vertex one, and ingoing at vertex two; this corresponds to the negatively charged vector boson being incoming at vertex one, and outgoing at vertex two. Treating the vertex Feynman-rules with incoming fields only, we thus get the coupling matrix of the negatively charged vector boson at vertex one, and that of the positively charged vector boson at vertex two. By adding the conjugate diagram—i.e., the one with reversed charge flow—we obtain the structure of the two coupled $G_{\pm}^T \,\kappa\, G_{\mp}$ terms.

We can then apply what we have found here to theories with gauged—i.e., local—flavor symmetries. In $U(1)$ theories, we only have one, neutral gauge boson, so we can (and will) only get a $G^T \kappa\, G$ term. However, if we are in larger, non-abelian gauge theories such as $SU(2)$ or $SU(3)$, we can choose a basis of charged gauge bosons—as for instance with the $W^{\pm}$ bosons in the Standard Model. Then, we also obtain the coupled $G_{\pm}^T \,\kappa\, G_{\mp}$ terms; note that we are not forced to choose a basis with such charged gauge bosons.

We may also think in terms of renormalizability to consider the possible theories these terms may exist in. This applies in particular to *massive* vector bosons. Namely, massive vector bosons are not renormalizable by power-counting due to a problematic term $\sim p^{\mu}\, p^{\nu}/m^2$ in the propagator that approaches a constant for infinite loop momenta,

$$\frac{i}{p^2 - m^2}\left(-g^{\mu\nu} + \frac{p^{\mu}\, p^{\nu}}{m^2}\right) \xrightarrow{p \longrightarrow \infty} \text{const.} \tag{II-2.47}$$

This makes diagrams involving such propagators badly divergent, rendering the theory non-renormalizable. The only way out of this predicament is via spontaneously broken gauge symmetries and the Higgs mechanism [37]. This is a known fact of QFT; however, an independent, neutral, massive vector boson is, in fact, renormalizable—this can for instance be shown using the Stückelberg mechanism—[40, 41]. By independent, we mean that the neutral vector boson is not a part of some multiplet, i.e., it is not related to other massive bosons in the way that, for instance, the Z boson is to the $W^{\pm}$ bosons. What this tells us is that for massive vector bosons, the *only* possibility to obtain $G_{\pm}^T \,\kappa\, G_{\mp}$ terms is via a *spontaneously broken, non-abelian, flavor-nonuniversal gauge-theory*!

For an independent, neutral, massive vector boson, we are not forced to assume a gauge theory; however, we need flavor-dependent interactions with the lepton-doublets to obtain a $G^T \kappa\, G$ term in the $\beta_{\mathcal{K}}$-function. In other words, for such terms to appear, we require that—for some reason—the massive vector boson interacts with the different generations of lepton-doublets differently. While adding such interactions to the Lagrangian would in principle be possible, it is rather ad-hoc, and from a theoretical standpoint unsatisfying—we would rather have some symmetry, some principle to explain this collection of interactions. Therefore, it would be more “natural”, in a sense, to also assume the independent, neutral, massive vector boson to arise through a spontaneously broken $U(1)$ gauge theory—or at least that the couplings are flavor-nonuniversal due to some other symmetry. Note that we may also have multiple such independent, neutral, massive vector bosons.

Massless vector bosons, on the other hand, are more straightforward. Since they only propagate two degrees of freedom, the description using a four-vector field *requires* them to be gauge bosons to preserve Lorentz invariance—this is not required for massive vector bosons, see for instance sections (5.3) and (5.9) of [31]. Thus, the *only* way to obtain $G^T\,\kappa\,G$ or $G_\pm^T\,\kappa\,G_\mp$ terms in the $\beta_\mathcal{K}$-function via *massless* vector bosons is for them to be gauge bosons.

Let us now put the previous statements together. The vector bosons that give rise to the novel terms in the $\beta_\mathcal{K}$-function are required to be gauge bosons that couple to the lepton-doublets. These interaction terms thus need to be gauge-invariant. Therefore, the lepton-doublets need to transform under the respective gauge group such that their couplings remain invariant. In other words, the theories which give rise to $G^T\,\kappa\,G$ or $G_\pm^T\,\kappa\,G\mp$ terms in the $\beta_\mathcal{K}$-function arise from *flavor-nonuniversal gauge theories*, and in particular from *flavor gauge theories.*

Let us emphasize at this point that there is a difference between these two: flavor-nonuniversal gauge theories are such where the *different lepton generations* transform under *different representations* of the gauge group; *however*, the local symmetry *does not need to be a flavor symmetry.* If, for instance, the first generation were in the triplet representation of $SU(2)_L$—for the sake of the argument we ignore all the problems this would incur—but the second and third generations remain doublets, this would constitute a flavor-nonuniversal gauge theory that is *not* a flavor gauge theory. Flavor gauge theories are such, where *lepton-flavor itself* constitutes the local symmetry. Thus, all flavor gauge theories are flavor-nonuniversal gauge theories, but not the other way around—flavor gauge theories are a *subset* of flavor-nonuniversal gauge theories.

Nevertheless, when discussing concrete examples, we will for the most part focus on flavor gauge theories. In particular, We will see specific examples of the points discussed here in $U(1)$ flavor gauge theories later on.

Let us summarize the discussion on G, and $G_\pm$ at his point, as these are the salient new terms discovered in this work.

The $\beta_{\mathcal{K}}$ constituent $\boldsymbol{G}$ arises from coupling to **neutral vector bosons**, while the $\boldsymbol{G_\pm}$ arise from coupling to **charged vector bosons**. At the **one-loop level**, this is the **only possibility**.

Since massless vector bosons are require to be gauge bosons by Lorentz invariance, and massive vector bosons by renormalizability, these terms **only arise** in **flavor-nonuniversal gauge theories**. In particular, they arise in **flavor gauge theories**. The only exception to this is given by an independent, neutral, massive vector boson, which does not need to be a gauge boson. However, invoking, e.g., a flavor gauge symmetry provides a natural framework to explain its differing couplings to the different lepton generations.

Thus, $\boldsymbol{G}$ arises from coupling to neutral gauge bosons, and can therefore be **present in any flavor-nonuniversal gauge theory**, starting from $U(1)$. In larger symmetry groups, such as $SU(N)$, the **neutral basis** for the gauge bosons also **gives rise to** $\boldsymbol{\sim G^T \kappa\, G}$ terms in $\beta_{\mathcal{K}}$, with **off-diagonal entries of** $\boldsymbol{G}$.

We may also take the complexified, **charged basis**—given by complex linear combinations of fields and their respective group generators. The remaining **neutral gauge bosons** have diagonal coupling matrices and give rise to $\boldsymbol{\sim G^T \kappa\, G}$ terms, where G is proportional to the respective **diagonal charge matrix**. Their neutrality is consistent with the hermiticity condition of G, $G^\dagger = G$. On the other hand, **charged gauge bosons** lead to $\boldsymbol{\sim G_\pm^T \kappa\, G_\mp}$ terms in $\beta_{\mathcal{K}}$. In other words, $\sim G_\pm^T \kappa\, G_\mp$ terms **cannot occur in abelian gauge theories, only in non-abelian ones**. $\boldsymbol{G_+}$ and $\boldsymbol{G_-}$ correspond to **coupling matrices of the charged vector bosons and their antiparticles**, which is consistent with the **conjugacy relation** $\boldsymbol{G_\pm^\dagger = G_\mp}$.

We summarize the possible one-loop topologies that can give rise to each of the α, P, G, and $G_\pm$ terms in tab. (II-2.1). We will build on this diagrammatic representation to discuss further properties of α, P, G, and $G_\pm$, using the numbers on the right-hand side of the table to refer to each of the topologies.

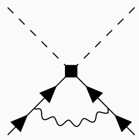
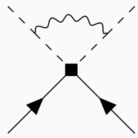
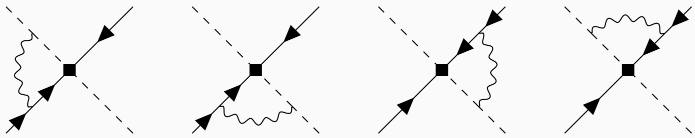
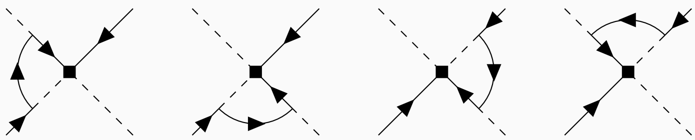
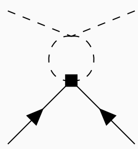
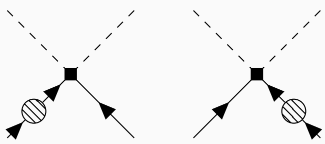
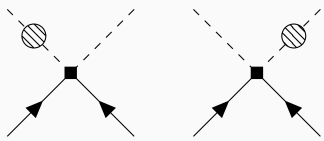
One-Loop Topology	$\boldsymbol{\alpha}$	$\boldsymbol{P}$	$\boldsymbol{G}$	#
	✓*	✗	✓	1.
	✓*	✗	✗	2.
	✓*	✓	✗	3.
	✓	✓*	✗	4.
	✓*	✗	✗	5.
	✓*	✓*	✗	6.
	✓*	✗	✗	7.

Table II-2.1.: Possible one-loop topologies that can give rise to each of the main terms of the $\beta_{\mathcal{K}}$-function; the terms are denoted by the respective matrices and scalars constituting them, where G represents both, G and $G_{\pm}$ matrices; the numbers on the right are used to refer to the different topologies in this table; the flows of the Higgs-doublets are left out for clarity, but are understood to be ingoing; the blobs summarize all diagrams contributing to the respective field renormalization; an asterisk ($*$) on checkmarks denotes contributions present in the Standard Model

II-2.1.3 Realness, Hermiticity, and Conjugacy Relations of the $\beta_{\mathcal{K}}$ Constituents

Let us now perform the remaining simplifications of α, P, G, and $G_\pm$ alluded to previously with the help of the diagrams in tab. (II-2.1). To that end, we will go through the different topologies of the rows one by one. We will show that all of them lead to hermitian, or in the case of flavor-scalars, real contributions to α, P, G, and $G_\pm$. We will thus show that

- α is real; note that α may be a matrix in a different space than lepton-flavor, in these spaces α is required to be symmetric by the symmetry of the Weinberg-Operator. Realness and symmetry also imply hermiticity. Thus,

$$\alpha^* = \alpha = \alpha^T = \alpha^\dagger\,; \tag{II-2.48}$$

- P is hermitian,

$$P^\dagger = P\,; \tag{II-2.49}$$

- G is hermitian,

$$G^\dagger = G\,; \tag{II-2.50}$$

- G_+ and G_- are conjugate to each other,

$$G_\pm^\dagger = G_\mp\,. \tag{II-2.51}$$

1. : In the first row, we see a vertex correction coming from vector bosons being exchanged between the two lepton-doublets. This interaction conserves lepton-number. For such an interaction, we can make the ansatz

$$\mathcal{L} \supset g_{ij}\,\overline{l^i}\,\gamma^\mu\,l^j\,V_\mu \quad (+\,\text{h.c.}) \tag{II-2.52}$$

for similar reasons as when discussing the form of the interaction form that can give rise to G. The coupling matrix g may in principle be complex. However, if V_μ is a real vector boson, then no hermitian conjugate is added, meaning that g must be a *hermitian* matrix. Note that even if we add a hermitian conjugate for a non-hermitian g, since the rest of the interaction term is hermitian, we would end up with the hermitian expression $g + g^\dagger$ in the Lagrangian—we could then just rename this as g. For a complex vector boson, we need the hermitian conjugate, and $g^\dagger \neq g$ in general. In the Feynman-rule for the vector interaction of eq. (II-2.52), we get an extra factor of i. For a real vector boson, this leaves us with an anti-hermitian vertex Feynman-rule. However, since we have the same coupling on both sides of $\mathcal{K}$, the two factors of i cancel.

From $\mathcal{K}$, we obtain one factor of i, which cancels with the i we get in the vertex Feynman-rule for $\delta\mathcal{K}$. This is true for all vertex corrections, so we will not mention this explicitly again. The loop-integral contains three propagators, giving us a total of four i factors—one from each propagator and one from the one-loop integral itself. This is true for all diagrams of topologies $1-4$, so we will not mention this explicitly for the remaining ones, either.

In total, all factors of i cancel, and we are left with a contribution that sandwiches κ between two hermitian matrices. This means for the $\beta_{\mathcal{K}}$-function that this contribution to G is *hermitian.* Note here again the point we previously discussed in relation to possible anti-hermiticity: we may, in principle, have an overall minus as prefactor, which we could absorb into G, turning it anti-hermitian; however, we show later by explicit calculation that this contribution does not have such a minus. A redefinition of the gauge generators will also not alter this result, as such a choice would imply a corresponding change in the interaction terms and Feynman-rules, which would cancel out overall. We can also understand this from the standpoint of reparametrization invariance, i.e., the invariance of the physical results under a change of parametrization of the fields we use in the Lagrangian. In this sense, the physical $\beta_{\mathcal{K}}$-function cannot depend on a reparametrization of intermediate particles as above. Therefore, we may drop such a $\pm$ prefactor while retaining full generality.

We can apply the same line of arguments for a neutral vector boson to a complex one as well. However, since we have two interaction terms—one involving V_μ and one involving $V_\mu^\dagger$—the coupling matrices to the two lepton-doublet legs are the hermitian conjugates of one another. Furthermore, we need to add the diagram with the vector boson going the opposite direction. This gives us exactly the structure of the $\beta_{\mathcal{K}}$-function with $G_\pm$.

Thus, we have shown that the contributions to G_+ and G_- are necessarily the *hermitian conjugates of one another*; or, in other words, they are *conjugate to each other.* Since this discussion encompasses all possible contributions to G and $G_\pm$, we have thus shown:

$$G^\dagger = G\,, \tag{II-2.53}$$

$$G_\pm^\dagger = G_\mp\,. \tag{II-2.54}$$

If the couplings are scalars in flavor-space, the hermiticity of G translates to the corresponding contribution to α being *real* (in flavor-space). Furthermore, the G^T and $G_\pm^T$ are directly multiplied with G and $G_\mp$, respectively, since they are proportional to the unit matrix in flavor-space and thus commute with κ. We recall that these terms may—while being scalars in flavor-space—be matrices in other spaces. Therefore, the realness of their contributions to α is not trivial. First, we note that they are symmetric in these other spaces by definition—given that their terms in $\beta_{\mathcal{K}}$ are. Then, we need to consider the theories that can give rise to these terms, and the forms of the corresponding contributions in non-flavor spaces. Here, we make use of the gained knowledge thus far, and consider the different cases one by one.

$U(1)$ gauge theory: In $U(1)$ gauge theories, we have a single, neutral gauge boson. By definition, the generators of $U(N)$ groups are hermitian—following the usual convention in physics contexts. Therefore, the diagonal entries of these generators are *real.* Furthermore, since we are in a $U(1)$ gauge theory, the generator is *diagonal*; this means that it is purely real, and thus

$$\left(G^T\,G\right)^* = G^T\,G \subset \alpha \quad ✓\,, \tag{II-2.55}$$

i.e., the contribution to α is also *real*.

Independent, neutral, massive vector boson: We recall that an independent, neutral, massive vector boson does not need to be a gauge boson. However, due to its neutrality, the interaction matrix has to be hermitian, as we have seen previously. Furthermore, we recall the Stückelberg mechanism, which provides a mass to an individual, neutral, massive vector boson by writing the massive vector as a $U(1)$ gauge boson and a scalar field,

$$V_\mu = A_\mu - \frac{1}{m}\,\partial_\mu B\,. \tag{II-2.56}$$

Here, A_μ is the new gauge field, and B is the scalar Stückelberg field. However, introducing an appropriate gauge fixing term for A_μ leads to the Stückelberg field decoupling from the gauge field [42]. Furthermore, if we assume an interaction of V_μ with fermions

$$\mathcal{L}_{int} \supset e\,\overline{\psi}\,\gamma^\mu\,\psi\,V_\mu\,, \tag{II-2.57}$$

as we do in our case here, a nonrenormalizable interaction term with the scalar B seems to emerge from the Stückelberg formulation—notice that plugging eq. (II-2.56) into eq. (II-2.57) yields an interaction with coupling $\sim 1/m$. However, via a suitable unitary transformation, this problematic interaction of B with a fermion current can be removed, leaving us with an interaction that looks like the gauge coupling to fermions in QED [40]. Correspondingly, the fermions need to transform under this new gauge group. Following our previous arguments for $U(1)$ gauge theories, this then leads to a real, diagonal coupling matrix; and thus a *real* contribution to α.

***SU(N)* gauge theory:** In the case of $SU(N)$ gauge theories, we once again have, by definition, hermitian generators. However, this is not enough to argue the realness of α. This is because the relevant quantity we need to consider is

$$\sum G^T_{SU(N)}\,G_{SU(N)} \quad \sim \quad \sum_{A=1}^{\dim\left(SU(N)\right)} \quad (T^A_R)^T\,T^A_R\,, \tag{II-2.58}$$

where T^A_R are the generators in the representation R. The problem is that for a general hermitian matrix,

$$\left(M^T\,M\right)^* \overset{M^\dagger = M}{=} M\,M^T \neq M^T\,M\,. \tag{II-2.59}$$

However, in choosing the basis for $SU(N)$ generators, the matrices are usually split into real and imaginary, hermitian matrices—this is the neutral basis. On the other hand, the charged basis is chosen such that we remain with purely real matrices—we combine the real and complex matrices with nonzero entries in the same positions to make two conjugate, real matrices:

$$T^{(\text{charged})}_\pm \equiv T^{(\text{neutral})}_{\text{real}} \pm i\cdot T^{(\text{neutral})}_{\text{imaginary}}\,. \tag{II-2.60}$$

Therefore, any real product of such matrices is yet again real, and thus the sum of these products is as well. In choosing the charged basis, however, we obtain the sum over both the remaining neutral, and the now-charged interactions:

$$\sum_{\substack{\text{neutral}\\ \text{gauge bosons}}} G^T_{SU(N)}\, G_{SU(N)} \quad + \frac{1}{2} \sum_{\substack{\text{charged}\\ \text{gauge bosons}}} \left(G^T_{SU(N),+}\, G_{SU(N),-} + G^T_{SU(N),-}\, G_{SU(N),+}\right). \tag{II-2.61}$$

The $G_{SU(N)}$ matrices are proportional to the real, diagonal generators, and thus only give us real contributions. In this charged basis, following our previous arguments, the $G_{SU(N),\pm}$ also only give real contributions as they are proportional to the real, off-diagonal generators defined earlier. Therefore, the entire expression and thus the contribution to α is *real*. Because of reparametrization invariance, the realness of the scalar contribution to α cannot depend on the choice of basis for intermediate particles, and thus this holds also in the neutral basis. Furthermore, since we did not assume any particular representation for the generators, this is true in *any* representation of the gauge group.

Let us illustrate this point in two ways: first, we show how the realness emerges in the fundamental representation of $SU(N)$, where we can make use of the completeness relation of the generators; second, we explicitly show that this is true also in another representation, the triplet representation of $SU(2)$.

In the fundamental representation of $SU(N)$, the generators fulfill the completeness relation,

$$\sum_A T^A_{ij}\, T^A_{kl} = \frac{1}{2}\left(\delta_{il}\,\delta_{jk} - \frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right), \tag{II-2.62}$$

which can be shown using the fact that the hermitian generators, together with the unit matrix, form a complete basis of hermitian $N \times N$ matrices [43]. As we have seen before, we need to consider

$$\sum_A \left(T^A\right)^T T^A\,, \tag{II-2.63}$$

which we can obtain from eq. (II-2.62) by contracting it with δ_{ik}. Thus, we find for the jl entry of eq. (II-2.63):

$$\sum_A \left(\left(T^A\right)^T T^A \right)_{jl} = \sum_A \left(\left(T^A\right)^T \right)_{jk} \left(T^A\right)_{kl} \tag{II-2.64}$$

$$= \sum_A \left(T^A\right)_{kj} \left(T^A\right)_{kl} \tag{II-2.65}$$

$$= \sum_A \delta_{ik} \left(T^A\right)_{ij} \left(T^A\right)_{kl} \tag{II-2.66}$$

$$\overset{(II-2.62)}{=} \delta_{ik} \cdot \frac{1}{2} \left(\delta_{il}\, \delta_{jk} - \frac{1}{N}\, \delta_{ij}\, \delta_{kl} \right) \tag{II-2.67}$$

$$= \frac{1}{2} \left(\delta_{jl} - \frac{1}{N}\, \delta_{jl} \right) \tag{II-2.68}$$

$$= \frac{N-1}{2N}\, \delta_{jl}\,. \tag{II-2.69}$$

So we see directly that in the fundamental representation, the contribution is, in fact, real; and even diagonal in this additional internal space.

Since this equation will be very useful later as well, let us emphasize this result. We will supplement this with some additional relevant comments. Note that the point concerning charged bases will be explained in detail later, but we already anticipate it here for completeness.

The **self-transposed sum over $SU(N)$ generators in the fundamental representation appearing in non-flavor spaces of the $\beta_{\mathcal{K}}$-function** is given by

$$\sum_A \left(T^A\right)^T T^A = \frac{N-1}{2N}\,\mathbb{1} \tag{II-2.70}$$

If all the generators are in the neutral basis, the sum runs from $A=1$ to $A = \dim\big(SU(N)\big) = N^2-1$. If they are in the charged basis, conjugate generators appear together, and with additional normalization factors. In the charged basis, the sum thus runs over the remaining neutral generators and the conjugate pairs.

Note that in the usual parametrization, the **charged basis leads to generator sums $\frac{1}{2}\left(T_+^T T_- + T_-^T T_+\right)$. The factor 1/2 arising here also motivates the normalization factor of 1/2 in eq. (II-2.21)—this provides a direct corrspondence between $G_\pm \sim T_\pm$ and $G \sim T$**.

It is important to note that since we **sum over self-transposed products of generators**, the result is in general **different from the quadratic Casimir**. For the fundamental representation, we have calculated and shown this explicitly. However, this can be seen in general from the fact that **real, hermitian generators are symmetric, while imaginary, hermitian generators are anti-symmetric**. This means that some terms in the sum get a minus sign compared to the quadratic Casimir, which is why the **self-transposed sum of generators is furthermore not necessarily proporional to the unit matrix**.

Let us now investigate the example of a triplet $SU(2)$ representation. This will illustrate how, even in non-fundamental representations, we obtain real contributions. Furthermore, we will see that while the contribution is real, it will *not* be diagonal. Therefore, this also provides a concrete example of how an additional gauge symmetry that is universal in flavor-space, could lead to a non-scalar α in a different space. In the three-dimensional representation of $SU(2)$, the neutral basis generators are given by

$$T_1^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\,,\quad T_2^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\,,\quad T_3^{(3)} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\,. \tag{II-2.71}$$

We can now perform the basis change and go to the charged basis via

$$T_+^{(3)} = T_1^{(3)} + i \cdot T_2^{(3)} = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{II-2.72}$$

$$T_-^{(3)} = T_1^{(3)} - i \cdot T_2^{(3)} = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{II-2.73}$$

If we now perform the corresponding sum of eq. (II-2.61) over the generators, we obtain

$$\left(T_3^{(3)}\right)^T T_3^{(3)} + \frac{1}{2} \left(T_+^{(3)}\right)^T T_-^{(3)} + \frac{1}{2} \left(T_-^{(3)}\right)^T T_+^{(3)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} , \tag{II-2.74}$$

where the factors of 1/2 in front of the coupled $T_\pm$ terms come from the Feynman-rules for the vertices—these involve an extra $1/\sqrt{2}$ for each $T_\pm$ generator. As we see, this contribution is not only *real*, but also *not proportional to* $\mathbb{1}$! Therefore, we see here explicitly how flavor-universal contributions from additional gauge symmetries can still lead to non-scalar α in other spaces.

Note also that while we have only discussed $SU(N)$ here, the same holds for $SO(N)$ as well, since the generators are real and symmetric.

With that, we have discussed all cases for vector bosons, and shown that if they have flavor-universal couplings, they give only real contributions to α. Therefore, this loop topology only gives *real contributions* to α overall.

II-2.1.3.1 Insertions of Two-Point Interactions in Vertex Corrections

The question may arise whether the couplings on both sides can be different from one another due to insertions of two-point interactions, e.g., via kinetic mixing or mass-insertions. If we consider a vertex correction diagram with such an insertion, we see that it is, in fact, finite. For the loop momentum $\ell \longrightarrow \infty$, the propagators decrease as

$$\text{Scalars:} \qquad \sim \mathcal{O}\left(\frac{1}{\ell^2}\right) , \tag{II-2.75}$$

$$\text{Fermions:} \qquad \sim \mathcal{O}\left(\frac{1}{\ell}\right) , \quad \text{and} \tag{II-2.76}$$

$$\text{Vector Bosons:} \qquad \sim \mathcal{O}\left(\frac{1}{\ell^2}\right) . \tag{II-2.77}$$

Since the vertex topologies $1-4$ contain at most two fermion propagators, and have three propagators in total, the diagrams diverge at most as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^4)$, i.e., logarithmically. Note that the factor of $\mathcal{O}(\ell^4)$ comes from the diverging integral measure. Diagrams with couplings between the scalar and a vector boson get an additional momentum factor from the interaction vertex. That is because we need to have a derivative ∂^μ in the interaction term, since no other vector—e.g, γ^μ—is available for scalars to make it Lorentz-invariant. This means that a scalar propagator with a coupling to a vector boson behaves as a fermion propagator in terms of power-counting. Therefore, the diagrams of topology number 3 also diverge as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^4)$. Furthermore, two scalar propagators together also diverge as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^4)$.

This means that any such diagram with at least one two-point insertion into the topologies $1-5$ is *finite by power-counting*, since it would lead to an additional propagator counting at least as $\mathcal{O}(1/\ell)$. However, the renormalization constants—and thus the $\beta_{\mathcal{K}}$-function—are extracted from the UV-divergences, meaning that such diagrams *cannot contribute* to the renormalization of $\mathcal{K}$. We visualize the fact that a two-point insertion leads to an additional propagator in fig. (II-2.4).

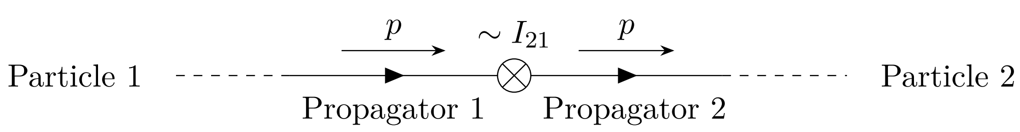


Figure II-2.4.: Insertion of a two-point interaction with coupling $\sim I_{21}$ turning particle 1 into particle 2; visualization of how a two-point insertion leads to an additional propagator

2. : In the second row, we see a vertex correction coming from vector bosons being exchanged between the two Higgs fields. We know that this type of coupling between a scalar and a vector field involves a derivative. The standard example of such an interaction term is given by a covariant derivative, generically following the form

$$\left[D_\mu \, \phi\right]^\dagger \left[D_\mu \, \phi\right] \supset -i \, g_\phi \, V_\mu \, \phi^\dagger \partial^\mu \, \phi + \text{h.c.}\,, \tag{II-2.78}$$

with

$$D_\mu = \partial_\mu + i \, g_\phi \, V_\mu \,. \tag{II-2.79}$$

g_ϕ is the coupling in flavor-space, which is scalar. Note, however, that it may be a tensor in another space—take, for instance, $SU(2)_L$ space with the hermitian $SU(2)_L$ generators. In the Feynman-rule for this interaction, we get another i, as well as an $\sim i \cdot p_\mu$, translating the derivative into the momentum p_μ. This means that, over all, we are left with an imaginary—or anti-hermitian—Feynman-rule. However, since we have two such couplings, the i factors cancel, and following the same line of arguments as for topology number 1, we thus know that this contribution to α is *real*.

3. : In the third row, we see vertex corrections coming from vector bosons being exchanged between one of the lepton-doublet legs, and one of the Higgs-doublet legs. Here, we can make use of the combined knowledge and arguments from the discussion of topologies 1 and 2. Note that these topologies can only contribute to the A matrix of $P = A + B\,C$ since their coupling to the flavor-singlet Higgs field cannot be a matrix in flavor-space. Let us now consider this combined interaction with one scalar and one lepton. For a real vector boson, since the imaginary—or possibly in a non-flavor-space, anti-hermitian—coupling with the Higgs is multiplied with i times the hermitian coupling matrix with the lepton-doublet, the factors of i cancel, and we are left with a *hermitian* matrix in flavor-space.

In the case of a complex vector boson, we again need to add the diagrams with its charge-flow going in both directions. For either of these, we have the product of the imaginary scalar coupling, and i times the in general non-hermitian coupling matrix to the lepton-doublet. From the reversed charge-flow, we need to add the hermitian conjugate of this product, thus leaving us with an overall *hermitian* expression—note that the two factors of i cancel. If the coupling of the vector bosons to the lepton-doublets is flavor-universal, then the hermiticity of the overall matrix we just proved, once again corresponds to a *real* scalar in flavor-space. This type of contribution enters, by definition, α instead of P. Note that since for a flavor-universal term from P, the matrix commutes with κ, this gives us the expression

$$\left(P^T + P\right) \kappa \, \subset \, \alpha \,. \tag{II-2.80}$$

This is by definition symmetric in the additional spaces, and as we have just shown, also hermitian. Combining these two properties, we see that this contribution is real:

$$\left(P^T + P\right)^* = P^\dagger + P^* \overset{P^\dagger = P}{=} P + P^T \quad \checkmark \,. \tag{II-2.81}$$

Therefore, the emerging contributions to α are *real in any internal space.*

4. : In the fourth row, we see vertex corrections coming from two Yukawa-coupling insertions, connecting one Higgs-doublet leg, and one lepton-doublet leg. We know that due to Lorentz-invariance, the intermediate fermion must be right-handed,

$$\mathcal{L} \supset g_{ij}^{Yuk.}\, \overline{f_R^i}\, l^j\, \phi + \text{ h.c.}\,, \tag{II-2.82}$$

where we neglect the specific $SU(2)_L$ structure needed for SM gauge-invariance. Note that, in principle, f_R does not need to carry lepton-flavor at this point in the consideration. We have proven before that two-point insertions cannot contribute to the $\beta_{\mathcal{K}}$-function via vertex correction diagrams. And since we need two external and internal Higgs fields, and the same for the lepton-doublets, we know that we need to involve the same fields at both vertices. Thus, the coupling matrices at both vertices need to be the same. To be precise, they need to be the hermitian conjugates of one another because of the fixed fermion-flow—at one vertex l is incoming and f_R outgoing, and at the other vice versa. Furthermore, the additional factors of i obtained from the Feynman-rules cancel. This means that their product, and thus the arising contributions to P, are *hermitian*,

$$\left[\left(g^{Yuk.}\right)^\dagger g^{Yuk.}\right]^\dagger = \left(g^{Yuk.}\right)^\dagger g^{Yuk.}\,. \tag{II-2.83}$$

As before, in the case of flavor-universal interactions, this yields a *real* contribution to α.
5. : In the fifth row, we see a vertex correction coming from a quartic Higgs self-coupling. This type of interaction is of the form

$$\mathcal{L} \supset \frac{\lambda}{2}\left|\phi^\dagger\,\phi\right|^2, \tag{II-2.84}$$

with a real coupling λ. Thus, the Feynman-rule is purely imaginary. Regarding the loop-integral, we get two factors of i from the propagators, and one from the integral itself. Therefore, all the factors of i cancel out, and we are left with a *real* contribution to α.

6. : In the sixth row, representatively, we have wave function renormalization diagrams of the lepton-doublets, which we visualize in fig. (II-2.5). We have shown at the beginning of this chapter that these contributions can only come from bubble diagrams, and can thus contribute to, e.g., the $B\,C$ term of $P = A + B\,C$, or the φ term of $\alpha = \varphi + \text{Tr}(\tilde{B}\,\tilde{C})$. For an incoming and outgoing lepton-doublet, we need to continue the fermion line throughout the diagram. This means that we can have fermion-number-conserving or -violating interactions with scalars, or fermion-number-conserving interactions with vectors.

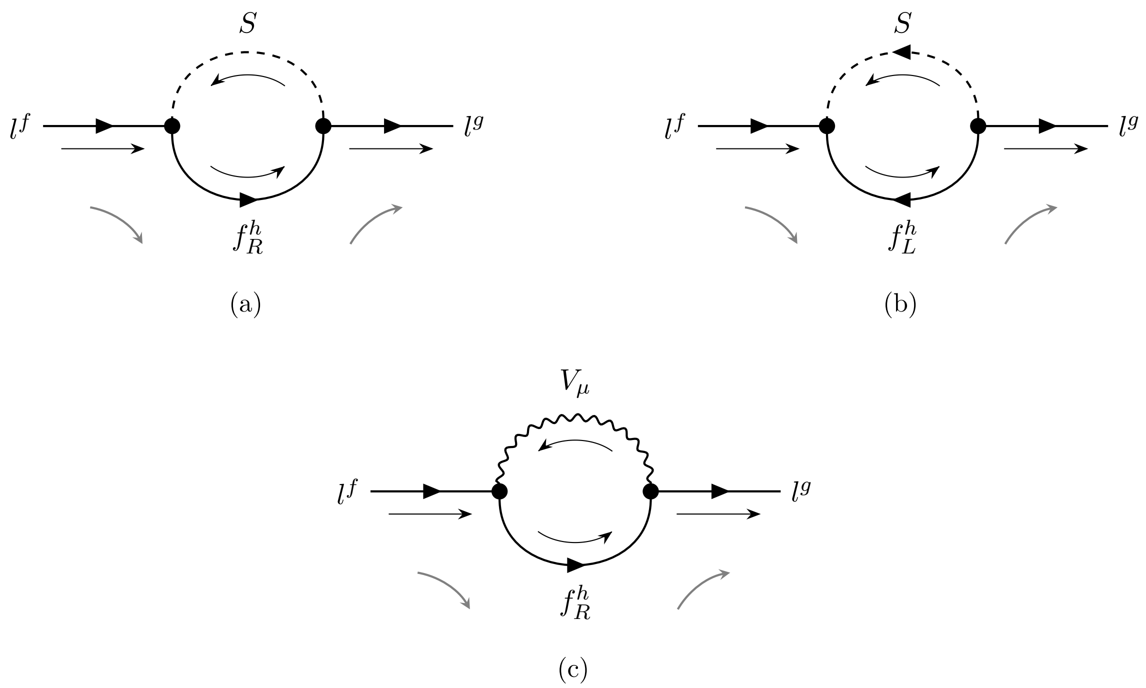


Figure II-2.5.: Generic wave function renormalization diagrams for the lepton-doublet due to some scalar S, vector V_μ, and fermions f_R^h and f_L^h; f_R^h and f_L^h are right-, and left-handed, respectively and may carry lepton-flavor h; fig. (II-2.5a) and fig. (II-2.5c) are due to fermion-number-conserving interactions with a scalar and vector, respectively; fig. (II-2.5b) is due to a fermion-number-violating interaction with a scalar; the gray arrows denote fermion-flow

We see directly from fig. (II-2.5) that since the fields involved in the two vertices of any diagram are the same, the vertices are as well. As before, to be precise, they are the hermitian conjugates of one another, meaning that their product is hermitian. This also holds for a complex vector boson—from the standpoint of the right vertex, a boson of the opposite charge is incoming, as compared to the left vertex. However, their coupling matrices are by definition just the hermitian conjugates of each other. Therefore, we obtain the same structure for complex vector bosons. However, we also need to sum over both flow directions, giving us either a sum of two manifestly hermitian terms, or equivalently, an anticommutator of the two coupling matrices. Factors of i from the vertex Feynman-rules cancel, as do the two propagators' factors of i. We are left with one i from the one-loop integral itself, which, however, cancels with the one from the Feynman-rule for the field-renormalization constant. Therefore, we have once again shown these contributions to P are *hermitian*; the contributions to α are real in flavor-space, and by the previously made symmetry arguments of eq. (II-2.81), *real in any space.*

With this, we have covered all possible one-loop topologies that can give rise to the P matrix, and shown that each of them leads to hermitian contributions. This means that we have shown that P *itself is hermitian*! Thus, we can re-cast the general form of P to

$$P = A + C^\dagger C + D_+^\dagger D_+ + D_-^\dagger D_- \,, \quad \text{with } A^\dagger = A \,, \tag{II-2.85}$$

$$= A + C^\dagger C + D_- D_+ + D_+ D_- \tag{II-2.86}$$

$$= A + C^\dagger C + \{D_+ \,, D_-\} \,, \tag{II-2.87}$$

where we have extended P by additional terms to explicitly include the fact that for some topologies, the hermitian terms can be rewritten as an anti-commutator. Note again that we have previously dropped the—in general—necessary sum over all possible terms with two flavor-dependent interaction vertices for simplicity.

7. : In the seventh row, representatively, we have wave function renormalization diagrams of the Higgs-doublets. We know that these can only contribute α, and that they are given by bubbles. The possible diagrams we can generically have are depicted in fig. (II-2.6). Since we can only have bubbles, and need an incoming and outgoing Higgs-doublet, there are four overarching, possible topologies. Namely, we can have two scalars, two vectors, one scalar and one vector boson, or two fermions in the loop. For the fermion-loop, we can have either two fermion-number-conserving, or -violating interactions.

We see from fig. (II-2.6) that the two vertices are once again the hermitian conjugates of one another, since the same fields are involved in both, but with conjugate flow. Therefore, these contributions to α are also *real* in flavor-space, and hermitian in other spaces. However, if we also take into account that α is symmetric in these additional spaces, we obtain as before,

$$\alpha^T = \alpha = \alpha^\dagger \tag{II-2.88}$$

$$\Longrightarrow \alpha^* = \alpha \,, \tag{II-2.89}$$

i.e., α is *real in internal spaces as well.*

Since this is the last possible contribution to α, we have thus shown that α *itself is real in any internal space*! Similarly to P, we can thus also parametrize it as

$$\alpha = \varphi + \mathrm{Tr}\left(\tilde{C}^\dagger \tilde{C}\right) + \mathrm{Tr}\left(\tilde{D}_+^\dagger \tilde{D}_+\right) + \mathrm{Tr}\left(\tilde{D}_-^\dagger \tilde{D}_-\right) , \quad \text{with } \varphi^T = \varphi^* = \varphi \,, \tag{II-2.90}$$

$$= \varphi + \mathrm{Tr}\left(\tilde{C}^\dagger \tilde{C}\right) + \mathrm{Tr}\left(\{\tilde{D}_+ \,, \tilde{D}_-\}\right) . \tag{II-2.91}$$

Note in particular that the trace over a hermitian matrix is by definition real because the entries on the diagonal are; furthermore, the trace of the transposed matrix is the same as of the matrix itself. Therefore, this parametrization is both *real and symmetric in any space.* We also note that we choose a parametrization analogous to that of P here, with the traces and symmetry condition on φ ensuring the transposition symmetry. However, we could also add the explicit transposed of φ, and of the matrices in the traces, which would be closer to the $P^T + P$ type of contribution to α—and in the case of φ also the contribution coming from vector boson vertex corrections. Nevertheless, for shorter notation, we do not do this here, and instead opt for the parametrization resembling P itself, as it still exhibits all necessary properties, and highlights the related origin. Note also that we could further simplify the expression via $\mathrm{Tr}\left(\{\tilde{D}_+ \,,\, \tilde{D}_-\}\right) = 2\ \mathrm{Tr}\left(\tilde{D}_+ \tilde{D}_-\right)$. Since they are ultimately of the same form as $\tilde{C}^\dagger \tilde{C}$, we will drop these terms for both α and P, however.

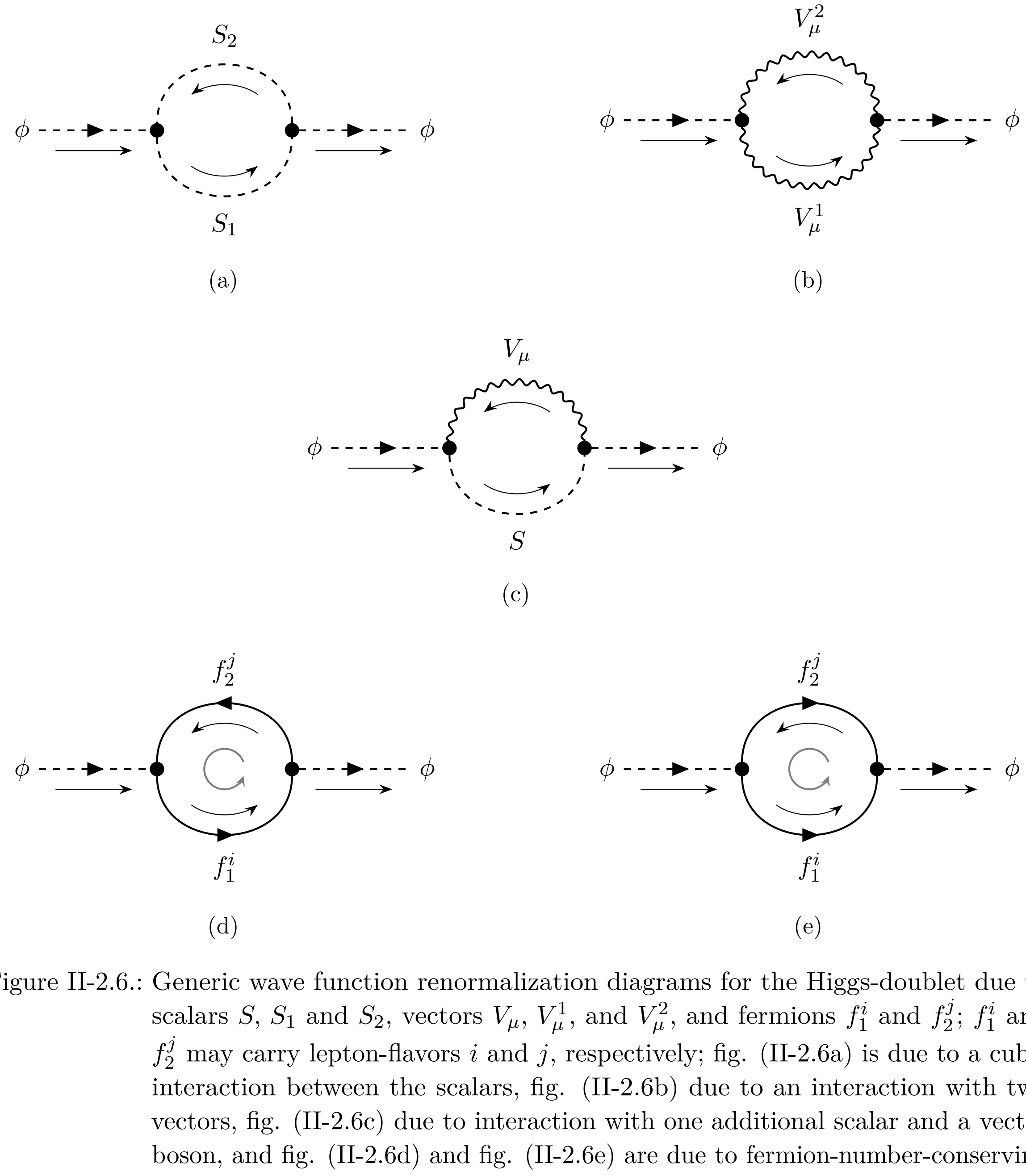


Figure II-2.6.: Generic wave function renormalization diagrams for the Higgs-doublet due to scalars S, S_1 and S_2, vectors V_μ, V_μ^1, and V_μ^2, and fermions f_1^i and f_2^j; f_1^i and f_2^j may carry lepton-flavors i and j, respectively; fig. (II-2.6a) is due to a cubic interaction between the scalars, fig. (II-2.6b) due to an interaction with two vectors, fig. (II-2.6c) due to interaction with one additional scalar and a vector boson, and fig. (II-2.6d) and fig. (II-2.6e) are due to fermion-number-conserving and -violating interactions with two fermions, respectively; the gray arrows denote fermion-flow

II-2.1.3.2 Insertions of Two-Point Interactions in Field-Renormalization Diagrams

Contrary to the vertex corrections, we *can* have two-point insertions in the bubble diagrams contributing to the field-renormalization of the lepton- or Higgs-doublets. Let us now show that such insertions are still compatible with the hermiticity and realness statements from above. We have two separate arguments we can use to show this:

- Reparametrization invariance of our theory, or
- Sum over one-particle-irreducible diagrams.

But first, let us perform the power-counting to see where exactly we may have such two-point insertions. For the lepton-doublet field renormalization, we have two propagators, one of which is a fermion propagator, and the other is either a scalar or vector boson propagator. This means that the diagrams diverge as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^3)$, i.e., linearly. Hence, a diagram with a two-point insertion on the scalar or vector boson propagators would be finite and thus not contribute to the renormalization of κ. Therefore, we can have at most one insertion on the fermion propagator, making the diagram diverge as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^4)$, i.e., logarithmically.

For the Higgs field-renormalization, the three diagrams with only scalar or vector boson propagators already diverge as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^4)$, so we cannot have any contributions coming from insertions. However, the fermion-loop diagrams diverge only as $\mathcal{O}(\ell^4) \cdot \mathcal{O}(1/\ell^2)$, i.e., quadratically. This means we can have up to two insertions over all. In principle, we may have one or two insertions on either propagator, or one on each.

First, let us argue using reparametrization invariance that this does not change our previous statements. Basically, the argument relies on the fact that we may redefine the basis of our fields in the Lagrangian. In the same way as we may choose a mass- or interaction-eigenbasis for particles—e.g., the lepton-doublets as we will do later on—we may choose an appropriate basis that diagonalizes the two-point interactions of the intermediate fermions. While this would lead to a redefinition of the coupling matrices, there would be no two-point interactions in the Lagrangian anymore and our previous arguments concerning the coupling structures hold. Since the physics is invariant under such Lagrangian-level field-redefinitions or -reparametrizations, the hermiticity of P and realness of α cannot depend on our choice of bases for the intermediate fields. Thus, if they are, respectively, hermitian and real in one basis, then they are necessarily hermitian in any basis, thus also in one where two-point interactions are present.

Second, let us argue using the fact that we need to sum over all possible one-particle-irreducible diagrams that can contribute to the process. For simplicity, let us take one two-point insertion on the bottom fermion propagator of the fermion-number-conserving interaction of fig. (II-2.6d) that renormalizes the Higgs field. However, while we discuss in detail only this case, the following arguments apply in the same way to the other possible diagrams with two-point insertions both for the Higgs- and the lepton-doublet as well. We show the corresponding diagram for this interaction in fig. (II-2.7).

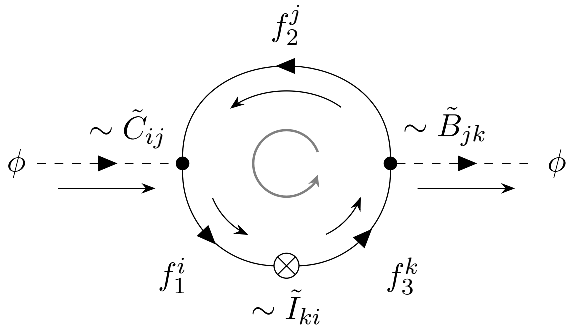


Figure II-2.7.: Generic wave function renormalization diagram for the Higgs-doublet due to some fermions f_1^i, f_2^j, and f_3^k that may carry lepton flavors i, j, and k, respectively; this diagram is due to some fermion-number-conserving interaction, and an insertion of a two-point coupling $\tilde{I}$ between f_1 and f_3; the gray arrow denotes fermion-flow

Inspecting the diagram of fig. (II-2.7), we see that the resulting contribution to α will be of the form

$$\alpha\Big|_{\text{insertion}} \sim \operatorname{Tr}\left(\tilde{B}\,\tilde{I}\,\tilde{C}\right), \tag{II-2.92}$$

since we can start at any of the vertices, and then go against the fermion-flow. By itself, this is not hermitian because in general, $\tilde{B} \neq \tilde{C}$ and $\tilde{I}^\dagger \neq \tilde{I}$. Note that for the field renormalization of the lepton-doublet, the only structural difference is that we do not have a trace over the interaction matrices. *However*, we need to keep in mind that we need to *sum* over *all* possible diagrams, which means that we also have the diagram with the reverse order of particles and vertices. That is, while the intermediate particles are still the same, we need to sum over all possible ways they can interact at the one-loop level, including the reverse way. In particular, we reverse the order of f_1^i and f_3^k. We show the thus corresponding "partner" diagram to fig. (II-2.7) in fig. (II-2.8).

From the diagram in fig. (II-2.8), we see that the contribution to α will have the form

$$\alpha\Big|_{\substack{\text{reverse}\\ \text{insertion}}} \sim \operatorname{Tr}\left(\tilde{C}^\dagger\,\tilde{I}^\dagger\,\tilde{B}^\dagger\right), \tag{II-2.93}$$

since the integral itself remains unchanged, and the interaction vertices present here are the hermitian conjugates of the ones appearing in fig. (II-2.7). Note that the conjugation appears due to the reverse ordering and charge flow of the particles. However, we see that eq. (II-2.93) is just the hermitian conjugate of the contribution from fig. (II-2.7) in eq. (II-2.92)! This means that when summing both of these contributions, the result is yet again hermitian, and via the symmetry of the trace *real*. This shows that even if we do not use the diagonal basis for intermediate fermions, the resulting possible insertions do not change the hermiticity of P, or realness of α.

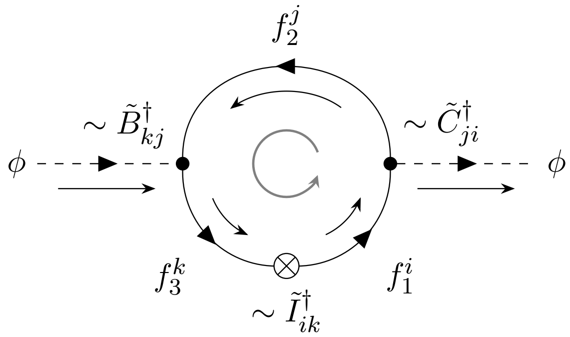


Figure II-2.8.: Generic wave function renormalization diagram for the Higgs-doublet due to some fermions f_1^i, f_2^j, and f_3^k that may carry lepton flavors i, j, and k, respectively; this diagram is due to some fermion-number-conserving interaction, and an insertion of a two-point coupling $\tilde{I}$ between f_1, and f_3; reverse diagram to fig. (II-2.7), reversing the order of intermediate particles f_1^i and f_3^k; the gray arrow denotes fermion-flow

The transformation behavior of the matrices arising from insertions is fixed by the surrounding interactions, but can in principle be both fermion-number conserving or violating, and the same for lepton-number. Note that, depending on the involved fermions, they may also be completely independent of lepton-flavor—i.e., be flavor scalars. Furthermore, the coupling matrices from the vertices with three fields could be the same, for example if mass and flavor-eigenbasis are different—in other words $f_1 = f_3$. In this example, we may, e.g., calculate the diagrams in the flavor-eigenbasis, and then have a mass insertion on one of the propagators. The sum of the two partner diagrams for this would lead to a hermitian contribution of the form

$$\mathrm{Tr}\left(\tilde{B}^\dagger\, \tilde{I}\, \tilde{B}\right) + \mathrm{Tr}\left(\tilde{B}^\dagger\, \tilde{I}^\dagger\, \tilde{B}\right), \tag{II-2.94}$$

which we could also rewrite as

$$\mathrm{Tr}\left(\tilde{B}^\dagger\, \tilde{I}_{herm.}\, \tilde{B}\right), \quad \text{with } \tilde{I}^\dagger_{herm.} = \tilde{I}_{herm.} = \tilde{I} + \tilde{I}^\dagger\,. \tag{II-2.95}$$

To include terms arising from insertions of two-point interactions, and thus present the expressions of α and P in full generality, we extend their initial forms. That is because the initial expressions—and the ones discussed outside the specific discussion on insertions—only considered vertices with three or more fields, which is usually the pertinent case. We thus summarize the findings of this section below. Note that we will change the naming of the sub-matrices to accommodate all terms and emphasize their distinction. However, the structure remains as before, and is only extended by additional sums over terms we dropped before, as well as the terms arising through insertions. If there are no insertions in the theory, many terms can be disregarded—these are the ones containing I_i or $\tilde{I}_i$ matrices.

The constituents $\boldsymbol{\alpha}$, $\boldsymbol{P}$, and $\boldsymbol{G}$ appearing in the one-loop $\beta_{\mathcal{K}}$-function are **hermitian**, and $\boldsymbol{G_\pm}$ are the **hermitian conjugates** to $\boldsymbol{G_\mp}$. Thus, these constituents satisfy

$$\alpha^* = \alpha = \alpha^T = \alpha^\dagger\,, \tag{II-2.96}$$

$$P^\dagger = P\,, \tag{II-2.97}$$

$$G^\dagger = G\,, \quad \text{and} \tag{II-2.98}$$

$$G_\pm^\dagger = G_\mp\,. \tag{II-2.99}$$

Furthermore, while $\boldsymbol{G}$ and $\boldsymbol{G_\pm}$ **cannot be decomposed further**, $\boldsymbol{\alpha}$ and $\boldsymbol{P}$ can be **decomposed into various terms arising from interactions involving at least three fields, as well as insertions of two-point interactions**. These decompositions involve **at most two vertices with three fields or more**, and **up to one or two insertions** for $\boldsymbol{P}$ and $\boldsymbol{\alpha}$, **respectively**. This decomposition is valid for **any number for lepton generations**. The general decomposition is given below, where the $\tilde{I}$ and I denote various insertions, the bracketed superscripts denote different matrices of the same form, and we define the flavor-space scalar φ. Note that the summation indices i denote matrices that appear with their hermitian conjugate, and the j denote those that do not—the corresponding appearing matrices are labeled by $\pm$ subscripts and are in general not the hermitian conjugates of each other. The tildes are present to distinuish the contributions to α and P, while highlighting similar structures. Note that, in practice, some of the matrices may or may not be the same.

We impose the conditions $\boldsymbol{A^\dagger = A}$ on the matrix A, and $\boldsymbol{\varphi^* = \varphi = \varphi^T = \varphi^\dagger}$ on the flavor-space scalar φ. As a result, **in full generality, at the one-loop level,** $\boldsymbol{P}$ **can thus be decomposed as**

$$\begin{aligned} P = A + \;& \sum_{i_B} \left(B^{(i_B)}\right)^\dagger B^{(i_B)} + \\ + \;& \sum_{i_C,\, i_{I_1}} \left(C^{(i_C)}\right)^\dagger \left\{\left(I_1^{(i_{I_1})}\right)^\dagger + I_1^{(i_{I_1})}\right\} C^{(i_C)} + \\ + \;& \left\{\sum_{j_C,\, i_{I_2}} \left(C_+^{(j_C)}\right)^\dagger I_2^{(i_{I_2})}\, C_-^{(j_C)} + \text{h.c.}\right\}. \end{aligned} \tag{II-2.100}$$

Furthermore, **in full generality, at the one-loop level, α can be decomposed as**

$$
\begin{aligned}
\alpha = \varphi + & \sum_{i_{\tilde{B}}} \mathrm{Tr}\left[\left(\tilde{B}^{(i_{\tilde{B}})}\right)^{\dagger} \tilde{B}^{(i_{\tilde{B}})}\right] + \\
+ & \sum_{i_{\tilde{C}}, i_{\tilde{I}_1}} \mathrm{Tr}\left[\left(\tilde{C}^{(i_{\tilde{C}})}\right)^{\dagger} \left\{\left(\tilde{I}_1^{(i_{\tilde{I}_1})}\right)^{\dagger} + \tilde{I}_1^{(i_{\tilde{I}_1})}\right\} \tilde{C}^{(i_{\tilde{C}})}\right] + \\
+ & \left\{\sum_{j_{\tilde{C}}, i_{\tilde{I}_2}} \mathrm{Tr}\left[\left(\tilde{C}_+^{(j_{\tilde{C}})}\right)^{\dagger} \tilde{I}_2^{(i_{\tilde{I}_2})} \tilde{C}_-^{(j_{\tilde{C}})}\right] + \text{h.c.}\right\} + \\
+ & \sum_{i_{\tilde{D}}, i_{\tilde{I}_3}} \mathrm{Tr}\left[\left(\tilde{D}^{(i_{\tilde{D}})}\right)^{\dagger} \left(\tilde{I}_3^{(i_{\tilde{I}_3})}\right)^{\dagger} \tilde{I}_3^{(i_{\tilde{I}_3})} \tilde{D}^{(i_{\tilde{D}})}\right] + \\
+ & \left\{\sum_{j_{\tilde{D}}, j_{\tilde{I}_3}} \mathrm{Tr}\left[\left(\tilde{D}_+^{(j_{\tilde{D}})}\right)^{\dagger} \left(\tilde{I}_{3,+}^{(j_{\tilde{I}_3})}\right)^{\dagger} \tilde{I}_{3,-}^{(j_{\tilde{I}_3})} \tilde{D}_-^{(j_{\tilde{D}})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{i_{\tilde{E}}, j_{\tilde{I}_4}} \mathrm{Tr}\left[\left(\tilde{E}^{(i_{\tilde{E}})}\right)^{\dagger} \left(\tilde{I}_{4,+}^{(j_{\tilde{I}_4})}\right)^{\dagger} \tilde{I}_{4,-}^{(j_{\tilde{I}_4})} \tilde{E}^{(i_{\tilde{E}})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{j_{\tilde{E}}, i_{\tilde{I}_4}} \mathrm{Tr}\left[\left(\tilde{E}_+^{(j_{\tilde{E}})}\right)^{\dagger} \left(\tilde{I}_4^{(i_{\tilde{I}_4})}\right)^{\dagger} \tilde{I}_4^{(j_{\tilde{I}_4})} \tilde{E}_-^{(j_{\tilde{E}})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{i_{\tilde{F}}, i_{\tilde{I}_5}} \mathrm{Tr}\left[\left(\tilde{F}^{(i_{\tilde{F}})}\right)^{\dagger} \left(\tilde{I}_5^{(i_{\tilde{I}_5})}\right)^{\dagger} \tilde{F}^{(i_{\tilde{F}})} \tilde{I}_5^{(i_{\tilde{I}_5})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{j_{\tilde{F}}, j_{\tilde{I}_5}} \mathrm{Tr}\left[\left(\tilde{F}_+^{(j_{\tilde{F}})}\right)^{\dagger} \left(\tilde{I}_{5,+}^{(j_{\tilde{I}_5})}\right)^{\dagger} \tilde{F}_-^{(j_{\tilde{F}})} \tilde{I}_{5,-}^{(j_{\tilde{I}_5})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{i_{\tilde{H}}, j_{\tilde{I}_6}} \mathrm{Tr}\left[\left(\tilde{H}^{(i_{\tilde{H}})}\right)^{\dagger} \left(\tilde{I}_{6,+}^{(j_{\tilde{I}_6})}\right)^{\dagger} \tilde{H}^{(i_{\tilde{H}})} \tilde{I}_{6,-}^{(j_{\tilde{I}_6})}\right] + \text{h.c.}\right\} + \\
+ & \left\{\sum_{j_{\tilde{H}}, i_{\tilde{I}_6}} \mathrm{Tr}\left[\left(\tilde{H}_+^{(j_{\tilde{H}})}\right)^{\dagger} \left(\tilde{I}_6^{(i_{\tilde{I}_6})}\right)^{\dagger} \tilde{H}_-^{(j_{\tilde{H}})} \tilde{I}_6^{(j_{\tilde{I}_6})}\right] + \text{h.c.}\right\} .
\end{aligned}
\tag{II-2.101}
$$

II-2.2 RGEs for the Eigenvalues of κ and the Leptonic Mixing Matrix

II-2.2.1 General Form

Previously, we have derived the most general, symmetry-allowed β_κ-function at the one-loop level and at order $1/\Lambda$. Now, let us turn our attention to the eigenvalues we can extract from κ, and the leptonic mixing matrix.

We can use the leptonic mixing matrix U to diagonalize κ. In the derivation of the RGEs for the eigenvalues, however, we need to keep in mind that U depends on the scale as well. To derive the most general RGEs for the eigenvalues, we will re-derive them from the start, since the derivation presented in other literature oftentimes takes a slightly different route. We will derive the RGEs in a general form and go into more detail concerning specific steps, which are not usually covered closely. In previous derivations, the specific form of β_κ is usually inserted before presenting a general formula valid for all explicit forms of β_κ. Therefore, we will derive these general equations here, and only at the very end plug in our expression for β_κ, given by eq. (II-2.21).

Recall that we have defined our diagonalization of κ via

$$\kappa \longrightarrow U^* \kappa U^\dagger = \kappa_{\text{diag}} \,, \tag{II-2.102}$$

and used the ansatz

$$\frac{\mathrm{d}U}{\mathrm{d}t} = T\,U \tag{II-2.103}$$

for the running of U. While it is usually not done explicitly, we can obtain the condition that T be anti-hermitian by calculating $\frac{\mathrm{d}}{\mathrm{d}t}\big(U\,U^\dagger\big)$. Let us verify this explicitly by using that $\frac{\mathrm{d}}{\mathrm{d}t}\big(U\,U^\dagger\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbb{1}\big) = 0$:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathbb{1}\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(U\,U^\dagger\big) \tag{II-2.104}$$

$$= \frac{\mathrm{d}U}{\mathrm{d}t}\,U^\dagger + U\,\frac{\mathrm{d}U^\dagger}{\mathrm{d}t} \tag{II-2.105}$$

$$= \frac{\mathrm{d}U}{\mathrm{d}t}\,U^\dagger + \left[\frac{\mathrm{d}U}{\mathrm{d}t}\,U^\dagger\right]^\dagger \tag{II-2.106}$$

$$= T\,U\,U^\dagger + \big[T\,U\,U^\dagger\big]^\dagger \tag{II-2.107}$$

$$= T + T^\dagger \tag{II-2.108}$$

$$\Longrightarrow T^\dagger = -T\,, \quad \text{i.e., } T \text{ is } \textit{anti-hermitian.}$$

We can now calculate $\frac{\mathrm{d}\kappa_{\mathrm{diag}}}{\mathrm{d}t}$ by making use of eq. (II-2.102):

$$\frac{\mathrm{d}\kappa_{\mathrm{diag}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\big(U^*\,\kappa\,U^\dagger\big) \tag{II-2.109}$$

$$= \underset{\text{insert } \mathbb{1} = U^T U^*}{\frac{\mathrm{d}U^*}{\mathrm{d}t}\uparrow}\kappa\,U^\dagger + U^*\,\frac{\mathrm{d}\kappa}{\mathrm{d}t}\,U^\dagger + U^*\,\kappa\,\underset{\text{insert } \mathbb{1} = U^\dagger U}{\uparrow\frac{\mathrm{d}U^\dagger}{\mathrm{d}t}} \tag{II-2.110}$$

$$= T^*\,\kappa_{\mathrm{diag}} + U^*\,\beta_\kappa\,U^\dagger + \kappa_{\mathrm{diag}}\,T^\dagger \tag{II-2.111}$$

$$\overset{T^\dagger=-T}{=} U^*\,\beta_\kappa\,U^\dagger + \big(T^*\,\kappa_{\mathrm{diag}} - \kappa_{\mathrm{diag}}\,T\big)\,. \tag{II-2.112}$$

In the next step, we demand that the eigenvalues κ_i be real. Recall that these are the entries of the diagonalized matrix κ_{diag}. This is sensible because the eigenvalues are supposed to represent masses, which are real quantities—while in other contexts, the imaginary part of complex masses in some sense corresponds to a decay-width, we will not get into this here. This realness condition leads us to the general RGEs for the mass eigenvalues in terms of the β_κ-function,

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} = \mathrm{Re}\left[U^*\,\frac{\mathrm{d}\kappa}{\mathrm{d}t}\,U^\dagger\right]_{ii} = \mathrm{Re}\Big[U^*\,\beta_\kappa\,U^\dagger\Big]_{ii} \qquad (\text{no sum})\,. \tag{II-2.113}$$

Here, we have used that

$$\mathrm{Re}\big(T^*\,\kappa_{\mathrm{diag}} - \kappa_{\mathrm{diag}}\,T\big)_{ii} = \kappa_i\cdot\mathrm{Re}\big(T^* - T\big)_{ii} = \kappa_i\cdot\mathrm{Re}(-2\,T_{ii}) = 0 \tag{II-2.114}$$

since T is anti-hermitian, i.e., its diagonal entries are purely imaginary,

$$\mathrm{Re}(T_{ii}) = 0\,. \tag{II-2.115}$$

We thus observe from eq. (II-2.113) that the RGEs of the eigenvalues of κ are given by the diagonal entries of the β_κ-function that is transformed with U.

On the other hand, we also obtain a condition on the T-matrix form the imaginary part,

$$0 \overset{!}{=} \mathrm{Im}\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} \tag{II-2.116}$$

$$= \mathrm{Im}\Big[U^*\,\beta_\kappa\,U^\dagger\Big]_{ii} + \big(T^*\,\kappa_{\mathrm{diag}} - \kappa_{\mathrm{diag}}\,T\big)_{ii} \tag{II-2.117}$$

$$= \mathrm{Im}\Big[U^*\,\beta_\kappa\,U^\dagger\Big]_{ii} + \kappa_i\cdot\mathrm{Re}\big(T^* - T\big)_{ii} \tag{II-2.118}$$

$$= \mathrm{Im}\Big[U^*\,\beta_\kappa\,U^\dagger\Big]_{ii} - 2\kappa_i\cdot\mathrm{Im}(T_{ii})\,. \quad (\text{no sum}) \tag{II-2.119}$$

This leads us to

$$\mathrm{Im}(T_{ii}) = \frac{1}{i}\, T_{ii} = \frac{1}{2\kappa_i}\, \mathrm{Im}\Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ii} \qquad \text{(no sum)}\,. \tag{II-2.120}$$

Next, we need to derive the conditions on the off-diagonal elements of T. For that, we use that κ_{diag} is by definition diagonal, i.e., $(\kappa_{\mathrm{diag}})_{ij} = 0$ for $i \neq j$. By plugging in eq. (II-2.111), we obtain the equation

$$0 \overset{i\neq j}{=} \Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij} + \big(T^*\, \kappa_{\mathrm{diag}} - \kappa_{\mathrm{diag}}\, T\big)_{ij} \tag{II-2.121}$$

$$= \Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij} + T^*_{ij}\, \kappa_j - \kappa_i\, T_{ij} \qquad \text{(no sum)}\,, \tag{II-2.122}$$

which is a *complex* equation. Therefore, we can take the real and imaginary part of this equation to obtain corresponding conditions on the real and imaginary parts of T_{ij}. Note that we obtain a minus sign for $\mathrm{Im}(T^*_{ij})$ with respect to $\mathrm{Im}(T_{ij})$ due to the anti-hermiticity of T. Thus, we obtain

$$0 \overset{i\neq j}{=} \mathrm{Re}\Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij} + (\kappa_j - \kappa_i)\, \mathrm{Re}(T_{ij})\,, \tag{II-2.123}$$

$$0 \overset{i\neq j}{=} \mathrm{Re}\Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij} - (\kappa_j + \kappa_i)\, \mathrm{Im}(T_{ij})\,. \tag{II-2.124}$$

Inverting these equations for the real and imaginary parts of T_{ij} gives us

$$\mathrm{Re}(T_{ij}) \overset{i\neq j}{=} \frac{1}{\kappa_i - \kappa_j}\, \mathrm{Re}\Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij}\,, \tag{II-2.125}$$

$$\mathrm{Im}(T_{ij}) \overset{i\neq j}{=} \frac{1}{\kappa_i + \kappa_j}\, \mathrm{Im}\Big[U^*\, \beta_{\kappa}\, U^\dagger\Big]_{ij}\,. \tag{II-2.126}$$

Let us now summarize our findings thus far. Note that we have not made *any* assumptions concerning the loop-order, order in $1/\Lambda$, structure or any other properties of the β_{κ}-function. All we have used is that the eigenvalues are real since they correspond to masses (after Electroweak Symmetry Breaking), and that, by definition, κ is diagonalized as $U^*\, \kappa\, U^\dagger = \kappa_{\mathrm{diag}}$.

The **RGEs** for the **eigenvalues** $\boldsymbol{\kappa_i}$ of κ are given by the following collection of equations. These hold for **any structure, number of lepton generations, loop-order, and order in 1/Λ of** $\boldsymbol{\beta_\kappa = \frac{d\kappa}{dt}}$. Requiring nothing more than that the κ_i be real, and $\kappa_{\mathrm{diag}} = U^* \kappa U^\dagger$, we obtain (without summing over repeated indices)

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} = \mathrm{Re}\Big[U^* \beta_\kappa U^\dagger\Big]_{ii} \tag{II-2.127}$$

The **evolution** of the **leptonic mixing matrix** $\boldsymbol{U}$ is given by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = T\,U\,, \quad \text{with } T \text{ anti-hermitian, and} \tag{II-2.128}$$

$$T_{ij} = \begin{cases} i \cdot \frac{1}{2\kappa_i}\,\mathrm{Im}\Big[U^* \beta_\kappa U^\dagger\Big]_{ii}\,, & \text{for } i = j\,, \\ \frac{1}{\kappa_i - \kappa_j}\,\mathrm{Re}\Big[U^* \beta_\kappa U^\dagger\Big]_{ij} + \frac{i}{\kappa_i + \kappa_j}\,\mathrm{Im}\Big[U^* \beta_\kappa U^\dagger\Big]_{ij}\,, & \text{for } i \neq j\,. \end{cases} \tag{II-2.129}$$

II-2.2.2 General Form with Explicit $\beta_{\mathcal{K}}$-Function

Starting from the expressions in eq. (II-2.127)–(II-2.129), we can now derive the corresponding expressions for our most general, symmetry-allowed, one-loop $\beta_{\mathcal{K}}$-function of eq. (II-2.21).

We start with the RGEs for the eigenvalues $\mathcal{K}_i$, plugging eq. (II-2.21) into eq. (II-2.127). Let us first derive an expression for the right-hand side of eq. (II-2.127), where we do not take the real part for now—i.e., we calculate $\left[U^* \beta_{\mathcal{K}} U^\dagger\right]_{ij}$ and ultimately set $i = j$. We will drop the summations over $G^{(r)}$ and $G_\pm^{(s)}$ for clarity, and reinstate them at the end. This gives us

$$\left[U^* \beta_{\mathcal{K}} U^\dagger\right]_{ij} = \Big[U^* \Big(\alpha \mathcal{K} + P^T \overset{\downarrow}{\ } \mathcal{K} + \mathcal{K} \overset{\downarrow}{\ } P + G^T \overset{\downarrow}{\ } \mathcal{K} \overset{\downarrow}{\ } G + \tag{II-2.130}$$

$$+ \frac{1}{2}\{G_+^T \overset{\downarrow}{\ } \mathcal{K} \overset{\downarrow}{\ } G_- + G_-^T \overset{\downarrow}{\ } \mathcal{K} \overset{\downarrow}{\ } G_+\}\Big) U^\dagger\Big]_{ij}$$

$$\overset{\substack{\text{insert}\\ \mathbb{1}\text{s}}}{=} \Big[\alpha \mathcal{K}_{\text{diag}} + U^* P^T U^T \mathcal{K}_{\text{diag}} + \mathcal{K}_{\text{diag}} \overbrace{U P U^\dagger}^{\equiv \tilde{P}} + U^* G^T U^T \mathcal{K}_{\text{diag}} \overbrace{U G U^\dagger}^{\equiv \tilde{G}} +$$

$$+ \frac{1}{2}\{U^* G_+^T U^T \mathcal{K}_{\text{diag}} \underbrace{U G_- U^\dagger}_{\equiv \tilde{G}_-} + U^* G_-^T U^T \mathcal{K}_{\text{diag}} \underbrace{U G_+ U^\dagger}_{\equiv \tilde{G}_+}\}\Big]_{ij} \tag{II-2.131}$$

$$= \Big[\alpha \mathcal{K}_{\text{diag}} + \tilde{P}^T \mathcal{K}_{\text{diag}} + \mathcal{K}_{\text{diag}} \tilde{P} + \tilde{G}^T \mathcal{K}_{\text{diag}} \tilde{G} + \tag{II-2.132}$$

$$+ \frac{1}{2}\{\tilde{G}_+^T \mathcal{K}_{\text{diag}} \tilde{G}_- + \tilde{G}_-^T \mathcal{K}_{\text{diag}} \tilde{G}_+\}\Big]_{ij}$$

$$= \alpha \mathcal{K}_i \delta_{ij} + (\tilde{P}^T)_{ij} \mathcal{K}_j + \mathcal{K}_i \tilde{P}_{ij} + \sum_{k=1}^{n} (\tilde{G}^T)_{ik} \mathcal{K}_k \tilde{G}_{kj} + \tag{II-2.133}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} (\tilde{G}_+^T)_{ik} \mathcal{K}_k \tilde{G}_{-,kj} + \frac{1}{2} \sum_{k=1}^{n} (\tilde{G}_-^T)_{ik} \mathcal{K}_k \tilde{G}_{+,kj}$$

$$= \alpha \mathcal{K}_i \delta_{ij} + \tilde{P}_{ij} \mathcal{K}_i + \tilde{P}_{ji} \mathcal{K}_j + \sum_{k=1}^{n} \tilde{G}_{ki} \tilde{G}_{kj} \mathcal{K}_k + \tag{II-2.134}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left(\tilde{G}_{+,ki} \tilde{G}_{-,kj} + \tilde{G}_{-,ki} \tilde{G}_{+,kj}\right) \mathcal{K}_k$$

$$\overset{i=j}{=} \underbrace{\alpha \mathcal{K}_i + 2 \tilde{P}_{ii} \mathcal{K}_i}_{\text{real}} + \sum_{k=1}^{n} \tilde{G}_{ki}^2 \mathcal{K}_k + 2 \frac{1}{2} \sum_{k=1}^{n} \tilde{G}_{+,ki} \tilde{G}_{-,ki} \mathcal{K}_k \,. \tag{II-2.135}$$

Here, we have defined n as the dimension of $\mathcal{K}$, i.e., the number of generations and correspondingly eigenvalues. Depending on the required, the $\mathbb{1}$s are inserted as $U^\dagger U$ or $U^T U^*$ at the arrows in the first line. Note that since P is hermitian, $P^\dagger = P$, the transformed $\tilde{P} = U\, P\, U^\dagger$ is as well. Thus, $\tilde{P}_{ii}$ is real due to $\tilde{P}_{ii} = \tilde{P}^\dagger_{ii} = \tilde{P}^*_{ii}$. Combined with the realness of α we showed in the previous section, this means that the first two terms in eq. (II-2.135) are purely real. Note also that the factors of 2 and 1/2 for $G_\pm$ cancel.

Similarly, we can also calculate $\left[U^*\, \beta_{\mathcal{K}}\, U^\dagger\right]_{ij}$ for $i \neq j$ to determine the off-diagonal entries of the T matrix. We may start from eq. (II-2.134) since we already performed some simplifications at that point, without explicitly taking the diagonal elements yet. Thus, we obtain for $i \neq j$:

$$\left[U^*\, \beta_{\mathcal{K}}\, U^\dagger\right]_{ij} \overset{(II-2.134),\; i\neq j}{=} \alpha\, \kappa_i\, \overset{0\,,\,\text{for } i \neq j}{\cancel{\delta_{ij}}} + \tilde{P}_{ij}\, \kappa_i + \tilde{P}_{ji}\, \kappa_j + \sum_{k=1}^{n} \tilde{G}_{ki}\, \tilde{G}_{kj}\, \kappa_k + \tag{II-2.136}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left(\tilde{G}_{+,\,ki}\, \tilde{G}_{-,\,kj} + \tilde{G}_{-,\,ki}\, \tilde{G}_{+,\,kj}\right) \kappa_k$$

$$\overset{\tilde{P}^\dagger = \tilde{P}}{=} \kappa_i\, \tilde{P}_{ij} + \kappa_j\, \tilde{P}^*_{ij} + \sum_{k=1}^{n} \tilde{G}_{ki}\, \tilde{G}_{kj}\, \kappa_k + \tag{II-2.137}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left(\tilde{G}_{+,\,ki}\, \tilde{G}_{-,\,kj} + \tilde{G}_{-,\,ki}\, \tilde{G}_{+,\,kj}\right) \kappa_k\,.$$

Unfortunately, at this level, we cannot simplify this further. However, we can make one more simplification for the real and imaginary parts of $\left[U^*\, \beta_{\mathcal{K}}\, U^\dagger\right]_{ij}$ regarding $\tilde{P}$ and $\tilde{P}^*$:

$$\left[U^*\, \beta_{\mathcal{K}}\, U^\dagger\right]_{ij} \overset{(II-2.137)}{=} \kappa_i\, \tilde{P}_{ij} + \kappa_j\, \tilde{P}^*_{ij} + \sum_{k=1}^{n} \left(\tilde{G}_{ki}\, \tilde{G}_{kj}\right) \kappa_k + \tag{II-2.138}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left(\tilde{G}_{+,\,ki}\, \tilde{G}_{-,\,kj} + \tilde{G}_{-,\,ki}\, \tilde{G}_{+,\,kj}\right) \kappa_k$$

$$= \left(\kappa_i + \kappa_j\right) \mathrm{Re}(\tilde{P}_{ij}) + i \cdot \left(\kappa_i - \kappa_j\right) \mathrm{Im}(\tilde{P}_{ij}) + \sum_{k=1}^{n} \tilde{G}_{ki}\, \tilde{G}_{kj}\, \kappa_k + \tag{II-2.139}$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left(\tilde{G}_{+,\,ki}\, \tilde{G}_{-,\,kj} + \tilde{G}_{-,\,ki}\, \tilde{G}_{+,\,kj}\right) \kappa_k\,.$$

With this, we are now in a position to finally write down the RGEs for the eigenvalues and T matrix in terms of the $\beta_{\mathcal{K}}$-function constituents. We simply need to insert eq. (II-2.135) and

(II-2.139) into eq. (II-2.127) and (II-2.129). We summarize these results below, reinstating the sum over arbitrary numbers of G and $G_{\pm}$ matrices. We see both in the results thus far, and in the following summary, that in the *mass eigenbasis*—i.e., considering the κ_i—the RGEs depend solely on the *transformed matrices* $\tilde{P}$, $\tilde{G}$, and $\tilde{G}_{\pm}$, which are defined precisely according to the transformation behavior described in the previous section. Finally, we obtain:

The **most general, symmetry-allowed RGEs for the eigenvalues $\boldsymbol{\kappa_i}$ of $\boldsymbol{\kappa}$, for any number of lepton generations $\boldsymbol{n}$, at the one-loop level, and at order $\boldsymbol{1/\Lambda}$** are given by the following collection of equations. The κ_i are real, and U is defined such that $\kappa_{\text{diag}} = U^* \kappa U^\dagger$. Defining the **transformed** matrices, $\tilde{P} = U\,P\,U^\dagger$ and $\tilde{G} = U\,G\,U^\dagger$ for brevity, we obtain for κ_i:

$$\frac{\mathrm{d}\kappa_i}{\mathrm{d}t} = \alpha\,\kappa_i + 2\,\tilde{P}_{ii}\,\kappa_i + \sum_r \sum_{k=1}^{n} \mathrm{Re}\Big[\big(\tilde{G}^{(r)}_{ki}\big)^2\Big]\,\kappa_k + \sum_s \sum_{k=1}^{n} \mathrm{Re}\Big[\tilde{G}^{(s)}_{+,ki}\,\tilde{G}^{(s)}_{-,ki}\Big]\,\kappa_k\,. \tag{II-2.140}$$

The **evolution** of the **leptonic mixing matrix $\boldsymbol{U}$** is given by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = T\,U\,, \quad \text{with } T \text{ anti-hermitian, and} \tag{II-2.141}$$

$$T_{ij} = \begin{cases} i\cdot\Big\{\sum_r \sum_{k=1}^{n} \mathrm{Im}\Big[\big(\tilde{G}^{(r)}_{ki}\big)^2\Big]\,\frac{\kappa_k}{2\,\kappa_i} + \sum_s \sum_{k=1}^{n} \mathrm{Im}\Big[\tilde{G}^{(s)}_{+,ki}\,\tilde{G}^{(s)}_{-,ki}\Big]\,\frac{\kappa_k}{2\,\kappa_i}\Big\}, & \text{for } i=j, \\[2ex] \frac{\kappa_i+\kappa_j}{\kappa_i-\kappa_j}\,\mathrm{Re}\big[\tilde{P}_{ij}\big] + i\cdot\frac{\kappa_i-\kappa_j}{\kappa_i+\kappa_j}\,\mathrm{Im}\big[\tilde{P}_{ij}\big] + \\ + \sum_r \sum_{k=1}^{n} \mathrm{Re}\Big[\tilde{G}^{(r)}_{ki}\,\tilde{G}^{(r)}_{kj}\Big]\,\frac{\kappa_k}{\kappa_i-\kappa_j} + \\ + i\cdot\sum_r \sum_{k=1}^{n} \mathrm{Im}\Big[\tilde{G}^{(r)}_{ki}\,\tilde{G}^{(r)}_{kj}\Big]\,\frac{\kappa_k}{\kappa_i+\kappa_j} + & \text{for } i\neq j. \\ + \sum_s \sum_{k=1}^{n} \mathrm{Re}\Big[\tilde{G}^{(s)}_{+,ki}\,\tilde{G}^{(s)}_{-,kj} + \tilde{G}^{(s)}_{-,ki}\,\tilde{G}^{(s)}_{+,kj}\Big]\,\frac{\kappa_k\,/\,2}{\kappa_i-\kappa_j} + \\ + i\cdot\sum_s \sum_{k=1}^{n} \mathrm{Im}\Big[\tilde{G}^{(s)}_{+,ki}\,\tilde{G}^{(s)}_{-,kj} + \tilde{G}^{(s)}_{-,ki}\,\tilde{G}^{(s)}_{+,kj}\Big]\,\frac{\kappa_k\,/\,2}{\kappa_i+\kappa_j}\,, \end{cases} \tag{II-2.142}$$

II-2.2.3 Discussion of the Eigenvalue and Leptonic Mixing Matrix RGEs

Let us start the eigenvalues' RGEs with a comment on their usage and computation. Namely, we would like to bring attention to how using the above equations (II-2.140)–(II-2.142) directly—e.g., in a numerical calculation—with zero or degenerate eigenvalues may lead to complications and divergences. This is because the formulae explicitly contain fractions with denominators made up of

- the eigenvalues themselves,
- differences of the eigenvalues, and
- sums of the eigenvalues.

Therefore, if we want to run the equations down from some scale Λ at which either

- one of the eigenvalues is zero,
- two eigenvalues are equal, or
- two eigenvalues are equal up to a minus sign,

we will divide by zero, which is evidently not ideal. This will lead to a divergence, which renders calculations difficult—in particular if this occurs at the boundary condition we start from. From a physical standpoint, such a divergence cannot be there, so where does it come from? After all, the original $\beta_{\mathcal{K}}$-function of eq. (II-2.21) did not have any parts that could diverge in this manner. The culprit lies in *dividing by*

- κ_i in eq. (II-2.119) to obtain eq. (II-2.120);
- $\kappa_i - \kappa_j$ in eq. (II-2.123) to obtain eq. (II-2.125); and
- $\kappa_i + \kappa_j$ in eq. (II-2.124) to obtain eq. (II-2.126);

which is an *illegal* step if these are zero. Therein also lies the solution, however. Namely, the numerators for the respective expressions are required to go to zero at least as quickly as the denominators, as dictated by the equations (II-2.119), (II-2.123), and (II-2.124). Nevertheless, the computational problem with such boundary conditions remains. In practice, it may therefore be advisable to consider the original $\beta_{\mathcal{K}}$-function itself, perform the RGE evolution, and diagonalize the resulting κ as required at the points of interest—or for numerous points along the RGE trajectory. This is, for instance, the method we used to compute the data of fig. (II-1.6), where we showed the RGE evolution of the three eigenvalues starting with only one non-zero value at some high scale.

Let us now take a closer look at eq. (II-2.140), dropping the summation over different contributions of the same type. As we have already seen in chapter (II-1), the RGEs of eq. (II-2.140) for each eigenvalue κ_i are, in general, *not* proportional to κ_i anymore if we have a $G \not\approx \mathbb{1}$ or $G_\pm \not\approx \mathbb{1}$. Why is this important? It is important because it is precisely this non-proportionality that we need to raise the rank of the mass matrix, i.e., generate new non-zero eigenvalues and neutrino masses. However, if the $G_{(\pm)}$ matrices are proportional to the unit matrix,

$$\tilde{G}_{(\pm)} = U\, G_{(\pm)}\, U^\dagger \sim U\, \mathbb{1}\, U^\dagger \tag{II-2.143}$$

$$\overset{U\,U^\dagger = \mathbb{1}}{=} \mathbb{1}\,. \tag{II-2.144}$$

Therefore, since the transformed quantities are then proportional to the unit matrix as well, their off-diagonal entries are zero, meaning that the terms $\sim \left(U\, G_{(\pm)}\, U^\dagger\right)_{ki} \kappa_k$ for $i \neq k$ vanish, rendering the RGEs once again proportional to κ_i—this is precisely what happens in the Standard Model. Therefore, to get one-loop eigenvalue RGEs that are not proportional to the eigenvalues themselves, *we need flavor-nonuniversal gauge theories*—let us re-state at this point that strictly speaking, there is one exception given by an individual, neutral, flavor-nonuniversally interacting, massive vector boson.

Next, we take a look at the sign of the $\sim G^T_{(\pm)}\, \kappa\, G_{(\mp)}$ contributions in eq. (II-2.140). We see that for general, hermitian G and conjugate $G_\pm$, the coefficients of κ_k can in principle be both positive *or* negative. Let us briefly check this. If G is a complex, hermitian matrix, $\tilde{G}$ is as well, even if U is real. Therefore, the coefficients $\sim \tilde{G}^2_{ki}$ are given by the real parts of squared complex numbers, which can very well be negative:

$$(a + i\, b)^2 = a^2 - b^2 + 2\, i\, a\, b \implies \text{real part negative for } a^2 < b^2. \tag{II-2.145}$$

Recall also that $G^\dagger_\pm = G_\mp$, and thus $\tilde{G}^\dagger_\pm = \tilde{G}_\mp$. This means that the coefficients depending on $\tilde{G}_\pm$ are given by the real part of

$$\tilde{G}_{+,\,ki}\, \tilde{G}_{-,\,ki} = \tilde{G}_{+,\,ki} \left(\tilde{G}^\dagger_+\right)_{ki} = \tilde{G}_{+,\,ki}\, \tilde{G}^*_{+,\,ik} \tag{II-2.146}$$

$$= \tilde{G}^*_{-,\,ik}\, \tilde{G}_{-,\,ki}\,, \tag{II-2.147}$$

which is also some positive or negative, real number. Therefore, we can induce eigenvalues of the same, or different sign in the general case. Since these eigenvalues ultimately give the neutrino masses, this may seem odd at first. In particular, this seems dangerous because an overall minus sign in the mass terms is characteristic of tachyons. However, this minus sign corresponds to a phase, and not negative mass squares. Therefore, inducing negative eigenvalues is not a problem. Furthermore, having positive and negative coefficients means that some running effects may partly compensate each other, i.e, that contributions coming from G and

$G_{\pm}$ may partially counteract each other. However, given that the mixing matrix runs as well, a broad cancellation across different scales seems unlikely. On the other hand, just as they may compensate each other, multiple same-sign contributions will accelerate the RGE evolution. This could mean, for instance, that larger eigenvalues can be attained, or fixed points are reached faster.

We can also consider the case with real $G_{(\pm)}$ matrices, and a real leptonic mixing matrix U. In this case, the $\tilde{G}_{(\pm)}$ are real as well, meaning that the coefficients coming from G are positive because they are squares of a real number! *However*, the ones coming from $G_{\pm}$ may still be negative! We see this by inspecting eq. (II-2.146) or (II-2.147), and noticing that even without the complex conjugation, we have the product of two real entries, which may be positive or negative. Thus, if we only have G matrices, but no $G_{\pm}$ matrices, the coefficients of these terms are all positive. This means that generated eigenvalues are of the same sign—assuming that the effect of potentially negative contributions from α and $\tilde{P}_{ii}$ do not overpower the ones from G.

Let us also recall here the previous discussion concerning the choice of basis or gauge bosons, and comment on its effect on the RGEs. If we are in a non-abelian gauge theory, we may choose a neutral or charged basis of the gauge bosons and their respective group generators. The complex basis is the one that leads to $G_{\pm}$ terms, but we may choose the neutral basis and only obtain G terms. However, this seems odd, given that the positivity of coefficients in eq. (II-2.140) depends on this choice. So then the following question arises:

II-2.2.3.1 Do the RGEs Depend on Our Choice of Basis for the Gauge Bosons?

The answer to this question is both *yes and no.* The *functional form* of the $\beta_{\mathcal{K}}$-function and eigenvalue RGEs evidently depends on the basis we choose; *however*, the *numerical values do not.* The reason for this is simply that choosing a different way to perform the calculation—charged or neutral basis of internal particles—cannot influence the physical effects of the RGE running, i.e., the numerical values of $\mathcal{K}(t)$, $\mathcal{K}_i(t)$, $\beta_{\mathcal{K}}(t)$, etc. This is again a result of reparametrization invariance of our theory. The critical point here is that while results for specific diagrams may be different, when summing all contributions to ultimately obtain the *physical* $\beta_{\mathcal{K}}$-function, any seemingly present differences between parametrizations must cancel. That is, our choice of basis for the fields cannot affect physical quantities and effects—in particular the running. Therefore, even if it appears that induced eigenvalues may now in principle be negative, any negative contribution cannot overpower the positive ones because we know from the neutral basis that this is not the case.

While we will do this more explicitly later when covering specific models, let us show an example of this already here to elucidate the point we made above. Consider the vertex correction giving rise to G and $G_{\pm}$ in, for instance, an $SU(N)$ flavor gauge theory. The details of this are not that important here, as we are only interested in seeing how the charged and neutral basis are equivalent and lead to the same RGEs, even if they initially seem different. Imagine for this sake two neutral gauge bosons, X_1 and X_2 with their respective hermitian group generators T_1 and T_2. We define the charged basis via a complex linear combination of the generators,

$$T_\pm \equiv T_1 \pm i\,T_2\,, \tag{II-2.148}$$

and then require

$$X_1\,T_1 + X_2\,T_2 \stackrel{!}{=} \frac{1}{\sqrt{2}}\,X_+\,T_+ + \frac{1}{\sqrt{2}}\,X_-\,T_-+\,, \tag{II-2.149}$$

where the prefactor of $1/\sqrt{2}$ is chosen for convenience. Eq. (II-2.149) is necessary to make sure we are using a valid reparametrization of the original fields. The factor of $1/\sqrt{2}$ ensures the correct normalization of

$$X_+\,X_- = \frac{1}{2}\,X_1^2 + \frac{1}{2}\,X_2^2 \tag{II-2.150}$$

for the complex fields $X_\pm$, and the real $X_{1/2}$. We can then determine the expression of $X_\pm$ in terms of $X_{1/2}$ from eq. (II-2.149), but we will not do that here, as it is not relevant for our current consideration. Note that we will go into more detail concerning this calculation in later chapters when discussing specific models. From eq. (II-2.149), we also see that the Feynman-rules involving $X_\pm$ instead of $X_{1/2}$ obtain an additional factor of $1/\sqrt{2}$ and use the generators $T_\pm$ instead of $T_{1/2}$. We show the diagrams for the vertex corrections due to $X_{1/2}$ and $X_\pm$ in fig. (II-2.9).

Since the four fields $X_\pm$ and $X_{1/2}$ are all vector bosons, the loop-integral itself is the same for all of them, the only difference lies in the different generators and their prefactors entering the Feynman-rules. Therefore, reading off the structure of the contributions from the diagrams is enough for our purposes here. Summing the respective contributions in both bases up, and dropping the universal factor from the loop-integral, we obtain the total contributions to the $\beta_{\mathcal{K}}$-function

$$\text{Neutral basis:} \qquad \beta_{\mathcal{K}} \sim T_1^T\,\kappa\,T_1 + T_2^T\,\kappa\,T_2\,, \quad \text{and} \tag{II-2.151}$$

$$\text{Charged basis:} \qquad \beta_{\mathcal{K}} \sim \frac{1}{2}\left(T_+^T\,\kappa\,T_- + T_-^T\,\kappa\,T_+\right). \tag{II-2.152}$$

First, notice that we can explicitly see the relations derived throughout this chapter emerge. The neutral basis leads to terms $\sim G_{1/2}^T\,\kappa\,G_{1/2}$, while the charged basis leads to coupled terms $\sim G_\pm^T\,\kappa\,G_\mp$. Furthermore, by definition, we know that $SU(N)$ generators are hermitian, and as expected $G_{1/2} \sim T_{1/2}$—therefore, since $T_{1/2}$ are hermitian, so are $G_{1/2}$. Lastly, we find explicitly the conjugacy relation $G_\pm^\dagger = G_\mp$ due to $G_\pm \sim T_\pm$, and the definition of $T_\pm$ according to eq. (II-2.148),

$$G_\pm^\dagger \sim T_\pm^\dagger = \left(T_1 \pm i\,T_2\right)^\dagger = T_1^\dagger \mp i\,T_2^\dagger = T_1 \mp i\,T_2 \tag{II-2.153}$$

$$= T_\mp \qquad \checkmark\,. \tag{II-2.154}$$

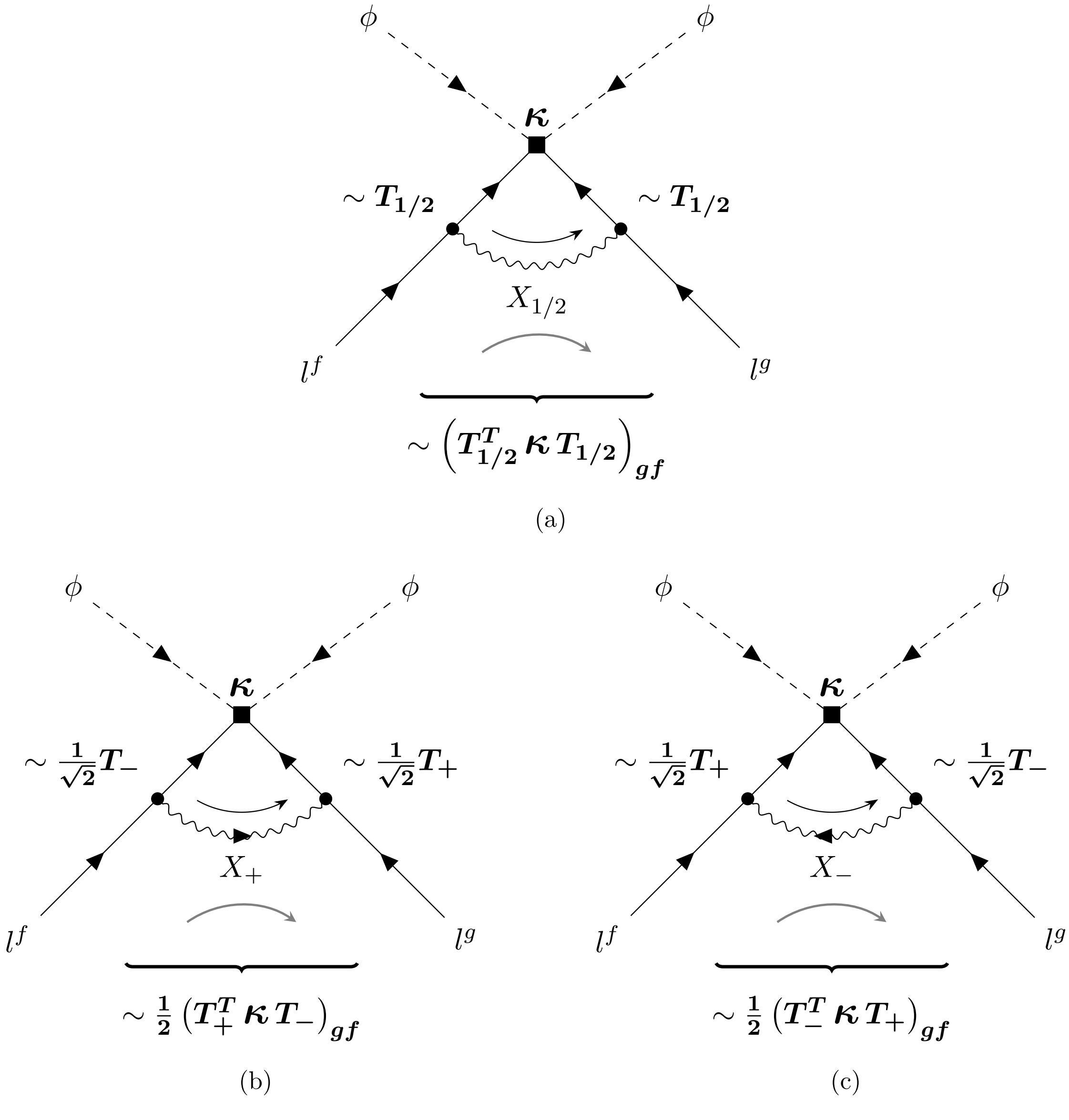


Figure II-2.9.: Structure of the $X\pm$ and $X_{1/2}$ contributions to the Weinberg-Operator via vertex correction; fig. (II-2.9b) shows the contribution coming from X_+, fig. (II-2.9c) that from X_-, and fig. (II-2.9a) that from X_1 and X_2; the gray arrow denotes fermion flow

We also see the factor of 1/2 emerge in the charged basis, hence the corresponding factor in the $G_\pm$ terms of $\beta_\mathcal{K}$. This gives us a direct correspondence between $G \sim T$ and $G_\pm \sim T_\pm$.

Returning to the $\beta_\mathcal{K}$-function, let us now verify that eq. (II-2.151) and eq. (II-2.152) do, in fact, give the same result, despite their different appearance. To that end, we plug the definition $T_\pm = \big(T_1 \pm i\,T_2\big)$ into eq. (II-2.152):

$$\frac{1}{2}\left(T_+^T\,\kappa\,T_- + T_-^T\,\kappa\,T_+\right) = \frac{1}{2}\Big[\big(T_1 + i\,T_2\big)^T\,\kappa\,\big(T_1 - i\,T_2\big)\; + \tag{II-2.155}$$

$$+\;\big(T_1 - i\,T_2\big)^T\,\kappa\,\big(T_1 + i\,T_2\big)\Big]$$

$$= \frac{1}{2}\Big[2\,T_1^T\,\kappa\,T_1 + 2\,T_2^T\,\kappa\,T_2\; + \tag{II-2.156}$$

$$+\; i\,T_2^T\,\kappa\,T_1\cdot(1-1) - i\,T_1^T\,\kappa\,T_2\cdot(1-1)\Big]$$

$$= T_1^T\,\kappa\,T_1 + T_2^T\,\kappa\,T_2 \qquad \checkmark\,. \tag{II-2.157}$$

As we see, even though the contributions from $X_\pm$ and $X_{1/2}$ look different in terms of their matrix structure in $\beta_\mathcal{K}$, fundamentally, *they give us the same result.* We have thus seen an explicit example, and personally verified that the physical $\beta_\mathcal{K}$-function does indeed not depend on the basis we choose for the intermediate particles; and that it gives the same physics independently of the basis-dictated matrix form it has.

Below, we summarize the main points made in the discussion concerning the calculation using the eigenvalue and mixing matrix RGEs, as well as the basis-independence of the $\beta_\mathcal{K}$-function.

The **RGEs** for the eigenvalues κ_i and the leptonic mixing matrix contain numerators that may **diverge in certain calculations**; specifically, if some eigenvalues are **zero** or **degenerate up to a sign**. This leads to **numerical divergences** in direct implementations with such boundary conditions. This is **not physical** and stems from a step in the derivation of the RGEs that is prohibited in these cases; physically, the numerators will go to zero at least as fast as the respective combination of eigenvalues in the denominators. This can be circumvented by **evolving the $\boldsymbol{\beta_{\mathcal{K}}}$ as a whole**, and **diagonalizing at specific points** to obtain $\boldsymbol{\kappa_i(t)}$.

The RGEs may **appear dependent on the choice of neutral or charged basis for the gauge bosons**, but this is **only superficial**. Any structural differences cancel out, and the **numerical values of $\boldsymbol{\beta_{\mathcal{K}}}$** are **independent of the chosen basis for intermediate particles**. The $\beta_{\mathcal{K}}$-function, as well as the eigenvalues κ_i are **physical quantities**, and as such, **do not depend on the parametrization of the calculation**. In practice, this is **due to the relation between neutral and charged bosons and their repsective group generators**.

We thus conclude this discussion here, and will move on to treat the in-practice calculation of the $\beta_{\mathcal{K}}$-function in the next section.

II-2.3 Calculating the β_κ-Function in Practice

Until now, we have discussed the general form of the β_κ-function and the origin and transformation behavior of the different parts and terms contained in it. In this section, we will cover the actual computation of the β_κ-function, and the renormalization constants it is made up of.

II-2.3.1 Calculating the β_κ-Function: Renormalization Constants

First, let us comment on some statements made in the previous sections. Namely, we frequently stated how the renormalization contributions affecting the Weinberg-Operator come from the vertex corrections and wave function renormalization. This is true independently of the model we are working in. This is because the bare coupling, as we have seen in eq. (I-3.60), depends solely on the renormalized coupling, and the renormalization constants we just mentioned:

$$\kappa_B = Z_\phi^{-\frac{1}{2}} \left(Z_l^T\right)^{-\frac{1}{2}} \left(\kappa + \delta\kappa\right) Z_l^{-\frac{1}{2}} Z_\phi^{-\frac{1}{2}} \,. \tag{II-2.158}$$

Therefore, the β_κ-function $\beta_\kappa = \mathrm{d}\kappa/\mathrm{d}t$ can only depend on the renormalization constants $\delta\kappa$, δZ_ϕ, and δZ_l—and the scale-dependent quantities within them. Therefore, the previous discussions hold, and any renormalization of κ stems from the vertex corrections and wave function renormalization.

However, we need to check the specific prefactors the renormalization constants enter the β_κ-function. In other words, we need to check whether, in full generality, the previously derived eq. (I-3.80),

$$\beta_\kappa = \delta\kappa_{,1} - \frac{1}{2}\left(\delta Z_{\phi,1} + \delta Z_{l,1}\right)^T \kappa - \frac{1}{2}\kappa \left(\delta Z_{\phi,1} + \delta Z_{l,1}\right), \tag{II-2.159}$$

holds at the one-loop level. For this, we can make use of our previous loop-topological discussions in section (II-2.1).

To prove this, let us first recall the general formula of eq. (I-3.59) in chapter (I-3),

$$\begin{aligned}
\beta_Q^{(0)} = {} & \left[D_Q \left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Q_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle - D_Q\, \delta Q_{,1} \right] + \\
& + Q \cdot \sum_{j\in J} n_j \left[D_Q \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_j,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] + \\
& + \sum_{i\in I} n_i \left[D_Q \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}Q} \middle| Q \right\rangle + \sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \right] \cdot Q \,.
\end{aligned} \tag{II-2.160}$$

Where, again, Q is the quantity whose β-function we are calculating, and δQ arises from additive renormalization of Q. The $\delta Z_{\phi_{i/j}}$ are defined via $Z = \mathbb{1} + \delta Z_{\phi_{i/j}}$ for every field $\phi_{i/j}$;

the fields with indices i and j are multiplied from the left and right on Q in the interaction term, respectively:

$$\mathcal{L} \sim \left(\prod_{i\in I} \phi_i^{2n_i}\right) Q \left(\prod_{j\in J} \phi_j^{2n_j}\right). \tag{II-2.161}$$

The $n_{i/j}$ are thus the powers of the renormalization constants entering the bare coupling Q_B:

$$Q_B = \left(\prod_{i\in I} Z_{\phi_i}^{n_i}\right) \left[Q + \delta Q\right] \mu^{D_Q\,(4-d)} \left(\prod_{j\in J} Z_{\phi_j}^{n_j}\right). \tag{II-2.162}$$

The V_A are quantities the renormalization constants depend on apart from Q—e.g., coupling constants. D_Q and the D_{V_A} are derived from the difference of the mass dimension of Q and V_A in d and 4 dimensions. These factors are defined via

$$[Q,\, V_A]_d = [Q,\, V_A]_{d=4} + D_{Q,\, V_A} \cdot (4-d) \tag{II-2.163}$$

$$\implies \quad D_{Q,\, V_A} = \frac{[Q,\, V_A]_d - [Q,\, V_A]_{d=4}}{4-d}. \tag{II-2.164}$$

Finally, let us also recall the definition of the products $\langle\,\cdot\,|\,\cdot\,\rangle$ for an arbitrary tensor F:

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \,\middle|\, y \right\rangle = \frac{\mathrm{d}F}{\mathrm{d}x}\, y \qquad \text{for scalars } x \text{ and } y, \tag{II-2.165}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \,\middle|\, y \right\rangle = \sum_m \frac{\mathrm{d}F}{\mathrm{d}x_m}\, y_m \qquad \text{for vectors } x \text{ and } y, \tag{II-2.166}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \,\middle|\, y \right\rangle = \sum_{m,\,n} \frac{\mathrm{d}F}{\mathrm{d}x_{m,\,n}}\, y_{m,\,n} \qquad \text{for matrices } x \text{ and } y, \text{ and} \tag{II-2.167}$$

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}x} \,\middle|\, y \right\rangle = \sum_{\substack{m_1,\,m_2,\\ m_3,\ldots}} \frac{\mathrm{d}F}{\mathrm{d}x_{m_1,\,m_2,\,m_3,\ldots}}\, y_{m_1,\,m_2,\,m_3,\ldots} \qquad \text{for arbitrary tensors } x \text{ and } y. \tag{II-2.168}$$

We also emphasize again that complex variables need to be considered as independent of their complex conjugates, i.e., we need to take derivatives etc. for both the variable and its complex conjugate.

Let us now check, and re-derive eq. (II-2.159) in full generality at the one-loop level. As previously alluded to, we will rely on our previous loop-topological discussions presented in section (II-2.1).

First, let us consider the possible interactions that can contribute to $\beta_{\mathcal{K}}$. This is necessary because we need to consider the dimensional scaling of each of the couplings, and in which powers they can appear—with this we can then straightforwardly employ eq. (II-2.160). The interactions we have seen are

1. Lepton gauge interactions: $\sim g_{1,\,ij}\,\overline{l^i}\,\gamma^\mu\,l^j\,V_\mu \quad (+\,\text{h.c.})$
   $\longrightarrow$ can contribute to δK, and δZ_l;
2. Higgs-vector interactions: $\sim -i\,g_2\,V_\mu\,S^\dagger \partial^\mu\,\phi\,+\,\text{h.c.}$
   $\longrightarrow$ can contribute to δK, and δZ_ϕ;
3. Yukawa interactions: $\sim g_{3,\,ij}\,\overline{f_1^i}\,f_2^j\,\phi\,+\,\text{h.c.}\,,$
   $\longrightarrow$ can contribute to δK, δZ_l, and δZ_ϕ;
4. Quartic Higgs interactions: $\sim \frac{\lambda}{2}\left|\phi^\dagger\,\phi\right|^2$
   $\longrightarrow$ can contribute to δK;
5. Cubic scalar interactions: $\sim g_5\,\phi\,S_1^\dagger\,S_2\,+\,\text{h.c.}$
   $\longrightarrow$ can contribute to δZ_ϕ;
6. Higgs-two-vector interactions: $\sim g_6\,\phi\,V_{1,\,\mu}\,V_2^\mu\,+\,\text{h.c.}$
   $\longrightarrow$ can contribute to δZ_ϕ; and
7. Fermion-fermion insertions: $\sim I_{ij}\,\overline{f_1^i}\,f_2^j\,+\,\text{h.c.}$
   $\longrightarrow$ can contribute to δZ_ϕ.

Note that we have named the fields according to our previous discussion, and that we have not listed fermion-number-violating interactions explicitly, as the dimensionality of the couplings is independent of (number-) flows. Furthermore, some fields may be the same even if they have different field symbols here—for instance, (2) includes Higgs gauge couplings. the purpose of this is to keep the terms as general as possible, within the confines of what is reasonable in this context. Now, we need to consider the scaling of their mass dimensions in dimensional regularization. To that end, let us recall the dimension of fundamental fields. We do this by inspecting the kinetic terms in the Lagrangian and use that the dimension of derivatives is fixed to $[\partial_\mu] = 1$. The Lagrangian has mass dimension d because the action $S = \int \mathrm{d}^d\,x\,\mathcal{L}$ is dimensionless, and $[\mathrm{d}\,x] = -1$; so we only need to subtract the number of derivatives in the kinetic terms from d, and divide by the number of respective fields:

- Scalars: $\mathcal{L}_{kin} \supset \partial_\mu\phi^\dagger\,\partial^\mu\phi \qquad \Longrightarrow [\phi] = \frac{1}{2}\cdot(d-2) = \frac{d-2}{2}$,
- Fermions: $\mathcal{L}_{kin} \supset \overline{\psi}\,\gamma^\mu\,\partial_\mu\psi \qquad \Longrightarrow [\psi] = \frac{1}{2}\cdot(d-1) = \frac{d-1}{2}$,

- Vectors: $\mathcal{L}_{kin} \supset F_{\mu\nu} F^{\mu\nu} \sim \left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 \quad \Longrightarrow [V_\mu] = \frac{1}{2} \cdot (d-2) = \frac{d-2}{2}.$

Next, we go through each of the interactions and determine the dimension of their couplings. we do this by requiring the entire interaction—as part of the Lagrangian—be of dimension d, and then subtracting the dimensions of derivatives and fields. For clarity, we will always subtract the dimensions in the order derivatives $\longrightarrow$ scalars $\longrightarrow$ fermions $\longrightarrow$ vectors in this calculation. By taking the limit $d \longrightarrow 4$, we obtain the dimension of the coupling in four spacetime dimensions, and by then subtracting these two and diving by $(4-d)$, we obtain the corresponding D_g—as described in eq. (II-2.164). Thus, we find:

① $$[g_1]_d = d - 0 - 0 \cdot \frac{d-2}{2} - 2 \cdot \frac{d-1}{2} - 1 \cdot \frac{d-2}{2} = \frac{4-d}{2} \quad \xrightarrow{d \longrightarrow 4} \quad [g_1]_{d=4} = 0$$

$$\Longrightarrow \quad D_{g_1} = \frac{\frac{4-d}{2} - 0}{4-d} = \frac{1}{2}\,,$$

② $$[g_2]_d = d - 1 - 2 \cdot \frac{d-2}{2} - 0 \cdot \frac{d-1}{2} - 1 \cdot \frac{d-2}{2} = \frac{4-d}{2} \quad \xrightarrow{d \longrightarrow 4} \quad [g_2]_{d=4} = 0$$

$$\Longrightarrow \quad D_{g_2} = \frac{\frac{4-d}{2} - 0}{4-d} = \frac{1}{2}\,,$$

③ $$[g_3]_d = d - 0 - 1 \cdot \frac{d-2}{2} - 2 \cdot \frac{d-1}{2} - 0 \cdot \frac{d-2}{2} = \frac{4-d}{2} \quad \xrightarrow{d \longrightarrow 4} \quad [g_3]_{d=4} = 0$$

$$\Longrightarrow \quad D_{g_3} = \frac{\frac{4-d}{2} - 0}{4-d} = \frac{1}{2}\,,$$

④ $$[\lambda]_d = d - 0 - 4 \cdot \frac{d-2}{2} - 0 \cdot \frac{d-1}{2} - 0 \cdot \frac{d-2}{2} = 4 - d \quad \xrightarrow{d \longrightarrow 4} \quad [\lambda]_{d=4} = 0$$

$$\Longrightarrow \quad D_\lambda = \frac{4-d-0}{4-d} = 1\,,$$

⑤ $$[g_5]_d = d - 0 - 3 \cdot \frac{d-2}{2} - 0 \cdot \frac{d-1}{2} - 0 \cdot \frac{d-2}{2} = \frac{6-d}{2} \quad \xrightarrow{d \longrightarrow 4} \quad [g_5]_{d=4} = 1$$

$$\Longrightarrow \quad D_{g_5} = \frac{\frac{6-d}{2} - 1}{4-d} = \frac{1}{2}\,,$$

⑥ $$[g_6]_d = d - 0 - 1 \cdot \frac{d-2}{2} - 0 \cdot \frac{d-1}{2} - 2 \cdot \frac{d-2}{2} = \frac{6-d}{2} \quad \xrightarrow{d \longrightarrow 4} \quad [g_6]_{d=4} = 1$$

$$\Longrightarrow \quad D_{g_6} = \frac{\frac{6-d}{2} - 1}{4-d} = \frac{1}{2}\,,$$

$$\textcircled{7} \quad [I]_d = d - 0 - 0 \cdot \frac{d-2}{2} - 2 \cdot \frac{d-1}{2} - 0 \cdot \frac{d-2}{2} = 1 \qquad \xrightarrow{d \longrightarrow 4} \qquad [I]_{d=4} = 1$$

$$\Longrightarrow \quad D_I = \frac{1-1}{4-d} = 0\,.$$

Let us also recall the above results for the Weinberg-Operator. Its interaction is of the form

$$\sim \kappa_{ij}\, l^i\, l^j\, \phi\, \phi\,, \tag{II-2.169}$$

and therefore we have that

$$\begin{aligned} &[\kappa]_d = d - 0 - 2 \cdot \frac{d-2}{2} - 2 \cdot \frac{d-1}{2} - 0 \cdot \frac{d-2}{2} = 3 - d \qquad \xrightarrow{d \longrightarrow 4} \qquad [\kappa]_{d=4} = -1 \\ &\Longrightarrow \quad D_\kappa = \frac{3 - d - (-1)}{4-d} = 1\,. \end{aligned} \tag{II-2.170}$$

Now, we consider the various loop-topologies, and in particular, the powers the respective couplings appear with. This is necessary to determine the values of the $\left\langle \frac{\mathrm{d}\delta Z_{\phi_i}}{\mathrm{d}V_A} \middle| V_A \right\rangle$ etc. The first step we can take is to consider κ itself—this corresponds to the first line of eq. (II-2.160) with the identification $Q = \kappa$.

We know that all vertex corrections are *linear* in κ, that $\kappa^\dagger$ does not appear, and that all wave function renormalization constants are *independent* of κ. Therefore, we find

$$D_\kappa \left\langle \frac{\mathrm{d}\delta\kappa_{,1}}{\mathrm{d}\kappa} \middle| \kappa \right\rangle = D_\kappa \cdot \sum_{m,n} \frac{\mathrm{d}\delta\kappa_{,1}}{\mathrm{d}\kappa_{mn}}\, \kappa_{mn} \overset{\text{linearity}}{=} D_\kappa \cdot \delta\kappa_{,1} \tag{II-2.171}$$

$$= \delta\kappa_{,1}\,. \tag{II-2.172}$$

This means that the first and last term of the first bracket in eq. (II-2.160) cancel, and we are left with

$$\sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta\kappa_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle\,. \tag{II-2.173}$$

Let us now determine this expression. Note that we will reference the topologies of tab. (II-2.1) again. We have shown in section (II-2.1) that we have exactly two vertices with three fields or more that can contribute to $\delta\kappa$, with the only exception of the quartic Higgs self-coupling. This encompasses the terms $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$, $\textcircled{5}$, and $\textcircled{6}$ from above. Note that some of these couplings also appear together in some diagrams—recall, e.g., topology number 3 in which a vector boson couples both to the Higgs and the lepton doublets. This means that we need to consider all terms and possible combinations the coupling matrices can appear in, including their hermitian conjugates. *However*, even if this sounds daunting, we can make use of a trick to simplify our consideration. To begin, we notice that all of these couplings have the same $D_{V_A} = 1/2$. Furthermore, due to the linearity of derivatives, we can consider each term in the sum of contributions individually. Moreover, terms that do not depend on the quantity we differentiate with regard to, drop out. Putting these factors together, our calculation becomes significantly easier. Namely, it is enough for us to calculate one general derivative expression that represents all coupling matrices. First, we define the collective D_{V_A} value for the pertinent interactions as $D_3 = 1/2$. Second, we denote all coupling matrices by V with some capital letter subscript to keep the notation general, and consistent with V_A. Third, we decompose the terms of $\delta\mathcal{K}_{,1}$ whose mass dimensions are characterized by D_3 as

$$\delta\mathcal{K}_{,1}\big|_{D_3} = \sum_{B,C}\sum_{i,j,k,l} \mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l)\, V_B^{ij}\, V_C^{kl}\,, \tag{II-2.174}$$

where we have factored out the matrix structure and $\mathcal{K}$ into $\mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l)$, and used that we have exactly two couplings. Finally, we apply the derivative, and plug in our previous findings:

$$\sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta\mathcal{K}_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle\Bigg|_{D_3} = D_3 \sum_{A,m,n} V_A^{mn} \frac{\mathrm{d}}{\mathrm{d}V_A^{mn}} \overbrace{\sum_{B,C}\sum_{i,j,k\,l} \mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l)\, V_B^{ij}\, V_C^{kl}}^{\delta\mathcal{K}_{,1}|_{D_3}} \tag{II-2.175}$$

$$= D_3 \sum_{B,C}\sum_{i,j,k\,l} \mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l) \sum_{A,m,n} V_A^{mn} \frac{\mathrm{d}}{\mathrm{d}V_A^{mn}} V_B^{ij}\, V_C^{kl} \tag{II-2.176}$$

$$= D_3 \sum_{B,C}\sum_{i,j,k\,l} \mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l) \quad \times \tag{II-2.177}$$

$$\times \sum_{A,m,n} \left[V_A^{mn}\, \delta_{AB}\, \delta^{mi}\, \delta^{nj}\, V_B^{kl} + V_A^{mn}\, V_A^{ij}\, \delta_{AC}\, \delta^{mk}\, \delta^{nl}\right]$$

$$= D_3 \sum_{B,C}\sum_{i,j,k\,l} \mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l) \quad 2\, V_B^{ij}\, V_C^{kl} \tag{II-2.178}$$

$$= 2\, D_3\, \delta\mathcal{K}_{,1}\big|_{D_3} \tag{II-2.179}$$

$$\overset{D_3=\frac{1}{2}}{=} \delta\mathcal{K}_{,1}\big|_{D_3}\,. \tag{II-2.180}$$

So the contributions coming from such vertex corrections are given directly by $\delta\mathcal{K}_{,1}\big|_{D_3}$ itself. The crucial point is that the specific combinations, matrix multiplication structure, etc. of V_B and V_B *are irrelevant here.* Since we factored out a completely general $\mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l)$, this applies to *all terms in the general RGE* of eq. (II-2.21), and *any sub-structure of its constituents.* That includes, for instance, the traces of α that would be given by $\mathcal{O}_{\mathcal{K}}^{D_3}(i,\,j,\,k,\,l) \sim \delta^{il}\,\delta^{jk}$, or unit matrices of flavor-scalars. Furthermore, since we kept the matrices fully general—summing over any B and C—this also applies to different coupling matrices, hermitian conjugates, squares etc.

We note that since the divergent parts of the vertex corrections cannot have any contributions coming from two-point insertions, the resulting vertex renormalization constant *cannot depend on any of the masses* that could be present in propagators. We recall that a massive propagator—consider the scalar propagator for simplicity—can be written as a geometric series of two-point insertions with massless propagators in-between:

$$\frac{i}{p^2-m^2} = \frac{i}{p^2}\,\frac{1}{1-\frac{m^2}{p^2}} \sim \frac{i}{p^2} + \frac{i}{p^2}\,(-i\,m^2)\,\frac{i}{p^2} + \tag{II-2.181}$$

$$+ \frac{i}{p^2}\,(-i\,m^2)\,\frac{i}{p^2}\,(-i\,m^2)\,\frac{i}{p^2} + \cdots . \tag{II-2.182}$$

Note that this is a schematic expansion to make the mass-independence clear, we do not consider technical details here.

Furthermore, we could also consider the gauge-fixing parameter ξ for gauge bosons. However, since these originate from Lagrangian terms

$$\mathcal{L}_{\text{gauge-fixing}} \sim -\frac{1}{2\xi_V}\,(\partial_\mu V^\mu)^2\,, \tag{II-2.183}$$

we find for ξ_V:

$$\begin{aligned} &-[\xi_V]_d = d-2-0\cdot\frac{d-2}{2} - 0\cdot\frac{d-1}{2} - 2\cdot\frac{d-2}{2} = 0 \qquad \xrightarrow{d\longrightarrow 4} \qquad [\xi_V]_{d=4} = 0 \\ &\Longrightarrow \quad D_{\xi_V} = \frac{0-0}{4-d} = 0\,. \end{aligned} \tag{II-2.184}$$

Note the negative sign before the mass dimension arises because ξ_V appears in the denominator in the Lagrangian. Therefore, any appearance of ξ_V in the contributions to $\delta\mathcal{K}_{,1}$ does not alter eq. (II-2.180), since the non-vanishing derivatives with respect to ξ_V are eliminated by the prefactor $D_{\xi_V} = 0$. Note that there are no terms where the ξ_V appear alone, as they necessarily always appear together with the gauge couplings—therefore no terms are lost, and we retain the full expression.

Before moving on to wave function renormalization constants, we need to inspect the effect of the quartic Higgs self-coupling. Here, we use that $\delta\mathcal{K}_{,1}$ is linear in this coupling, meaning we find straightforwardly:

$$\sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta\kappa_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle \Bigg|_\lambda = D_\lambda\, \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \overbrace{\text{const.}\cdot\lambda\,\kappa}^{\delta\kappa_{,1}|_\lambda} \tag{II-2.185}$$

$$= D_\lambda \text{ const.}\cdot\lambda\,\kappa \tag{II-2.186}$$

$$\stackrel{D_\lambda=1}{=} \delta\kappa_{,1}\big|_\lambda\,. \tag{II-2.187}$$

Therefore, also the term coming from the quartic Higgs self-coupling contributes exactly with its contribution to the vertex correction. And hence, *all* terms contribute exactly as they appear in $\delta\kappa_{,1}$, i.e.,

$$\sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta\kappa_{,1}}{\mathrm{d}V_A} \middle| V_A \right\rangle = \delta\kappa_{,1}\,. \tag{II-2.188}$$

This means that, at the one-loop level, the entire first bracket in eq. (II-2.160) is given by $\delta\kappa_{,1}$ *in full generality.*

Next, we discuss the wave function renormalization constants.

We have seen in the previous sections that κ does not appear in wave function renormalization, therefore

$$D_\kappa \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}\kappa} \middle| \kappa \right\rangle = D_\kappa \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}\kappa} \middle| \kappa \right\rangle = 0, \tag{II-2.189}$$

meaning that, once again, only the derivatives with respect to the V_A remain. Furthermore, we have argued previously that only bubble diagrams contribute to wave function renormalization in this context. This means that we can apply the exact same arguments as for $\delta\kappa_{,1}\big|_{D_3}$ also here. The only additional point we need to consider is the appearance of two-point insertions. As before, we can on one hand argue with reparametrization invariance. On the other hand, we can also follow a more direct approach. Namely, in the same way as with the gauge parameter ξ_V, since $D_I = 0$, no derivatives with respect to these insertions can contribute, as they are killed by the prefactor:

$$D_I \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}I} \middle| I \right\rangle = D_I \left\langle \frac{\mathrm{d}\delta Z_{\phi_i,1}}{\mathrm{d}I} \middle| I \right\rangle \stackrel{D_I=0}{=} 0\,. \tag{II-2.190}$$

Therefore, they only appear in the terms with derivatives with respect to the other interaction vertices—note again that no terms are lost, since the insertions only appear together with other vertices. Thus, their presence does not change which terms contribute and with which prefactors.

In wave function renormalization constants, mass dependencies can enter because the appearance of masses is equivalent to a two-point insertion coupling a field to itself. Therefore, following the same arguments as for insertions, masses do not contribute with separate terms, but only together with the other interactions. Therefore, we can absorb insertions and masses into the prefactor of V_B and V_C in a decomposition analogous to that of $\delta\mathcal{K}_{,1}$ in eq. (II-2.174):

$$\delta Z_{\phi_{i/j},1} = \sum_{B,C} \sum_{i,j,k,l} \mathcal{O}_{Z_\phi i/j}(i,\, j,\, k,\, l)\, V_B^{ij}\, V_C^{kl}\,, \tag{II-2.191}$$

where the V_B and V_C may or may not be the same as for $\delta\mathcal{K}_{,1}$. We can now apply the same calculation as before to reach the analogous result to eq. (II-2.180),

$$\sum_A D_{V_A} \left\langle \frac{\mathrm{d}\delta Z_{\phi_{i/j},1}}{\mathrm{d}V_A} \middle| V_A \right\rangle = \delta Z_{\phi_{i/j},1}\,. \tag{II-2.192}$$

The crucial point here again is that only the derivatives with prefactors D_3 contribute.

Therefore, we have proven that the second and third bracket contribute with exactly $\delta Z_{\phi_j,1}$ and $\delta Z_{\phi_i,1}$, respectively! From the form of the bare Weinberg-Operator in eq. (II-2.158), we can directly read off that

$$\phi_{i=1} = \phi\,, \tag{II-2.193}$$

$$\phi_{i=2} = l\,, \tag{II-2.194}$$

$$\phi_{j=1} = l\,, \tag{II-2.195}$$

$$\phi_{j=2} = \phi\,; \tag{II-2.196}$$

$$n_{i=1} = -\frac{1}{2}\,, \tag{II-2.197}$$

$$n_{i=2} = -\frac{1}{2}\,, \tag{II-2.198}$$

$$n_{j=1} = -\frac{1}{2}\,, \quad \text{and} \tag{II-2.199}$$

$$n_{j=2} = -\frac{1}{2}\,, \tag{II-2.200}$$

as we have also seen in chapter (I-3). Note that the ϕ without subscript is the Higgs field, while the ones with a subscript are the ones from eq. (II-2.161).

Now we can write down the final one-loop result, verifying eq. (II-2.159) *in full generality*. Putting our findings together, we thus obtain from eq. (II-2.160):

The $\boldsymbol{\beta_{\kappa}}$**-function** is **in full generality, for any number of lepton generations, any extension of the Standard Model, at the one-loop level, and to order $\mathbf{1/\Lambda}$** given by

$$\beta_{\kappa} = \delta\,\kappa_{,\,1} - \frac{1}{2}\left(\delta Z_{\phi,\,1} + \delta Z_{l,\,1}\right)^{T}\kappa - \frac{1}{2}\,\kappa\left(\delta Z_{\phi,\,1} + \delta Z_{l,\,1}\right). \quad \text{(II-2.201)}$$

The subscript 1 of the vertex renormalization constant $\delta\kappa_{,\,1}$, and the field renormalization constants $\delta Z_{\phi,\,1}$ and $\delta Z_{l,\,1}$ indicates that the quantities appear without the factor $1/(4-d)$ from the UV divergence.

Furthermore, $\boldsymbol{\delta\kappa_{,\,1}}$ **is independent of any masses of intermediate particles**.

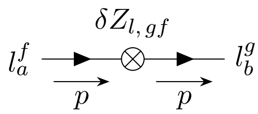
II-2.3.2 Renormalization Constants from Feynman-Diagrams

Now that we know how the renormalization constants enter the $\beta_{\mathcal{K}}$-function, let us briefly cover how we obtain them from the UV-divergences of pertinent diagrams. To begin, let us recall the form of the counterterms containing the renormalization constants. These have the same form as the kinetic terms, in particular they are

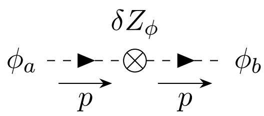
$$\mathcal{L}_{tot} \supset \mathcal{C}_{Z_l} = \delta Z_{l,\,gf}\, \overline{l^g}\, i\, \gamma^\mu\, \partial_\mu l^f \tag{II-2.202}$$

$$\mathcal{L}_{tot} \supset \mathcal{C}_{Z_\phi} = \delta Z_\phi \left(\partial^\mu \phi\right)^\dagger \partial_\mu \phi\,. \tag{II-2.203}$$

Using $\partial_\mu \longrightarrow -i\, p_\mu$ for incoming momenta, these lead to counterterm diagrams with Feynman-rules given by



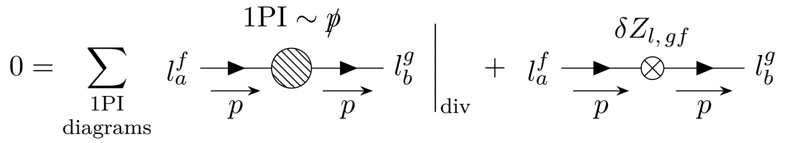
$$= i\, \not{p}\, \delta Z_{l,\,gf}\, P_L\, \delta_{ba}\,, \quad \text{and} \tag{II-2.204}$$



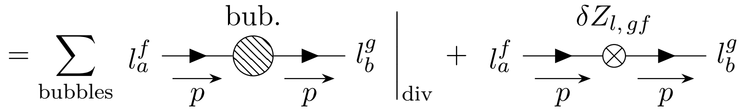
$$= i\, p^2\, \delta Z_\phi\, \delta_{ba}\,. \tag{II-2.205}$$

Note that the subscripts are $SU(2)_L$ indices, and we explicitly extracted the left-projector for the lepton-doublets. Furthermore, there are no dimensional rescaling factors, as the wave function renormalization constants are dimensionless by definition—the corresponding $D_{\delta Z}$ would turn out to be zero since the kinetic term is already of dimension four.

The counterterms now have to cancel all UV-divergences $\sim \not{p}$ for the lepton doublet self-energy diagrams, and $\sim p^2$ for the Higgs-doublet ones. This means that, by definition, the sum of the UV-divergent parts of all relevant one-particle-irreducible (1PI) diagrams and the counterterms has to vanish. However, we know from the previous sections that these diagrams are bubble diagrams—recall that we argued that only bubbles can contribute to wave function renormalization at the one-loop level. Therefore, we only need to consider bubble diagrams, which we will denote these by $\mathcal{B}_l$ for the lepton-doublet, and $\mathcal{B}_\phi$ for the Higgs-doublet. Diagrammatically, this corresponds to



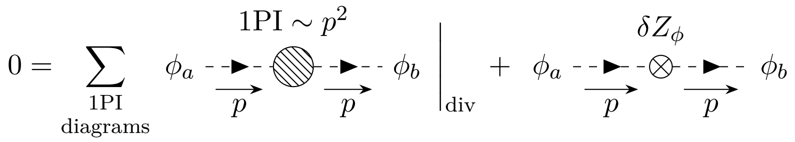
(II-2.206)



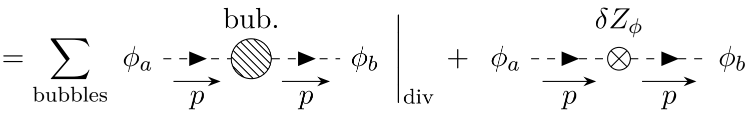
(II-2.207)

$$= \sum_i \mathcal{B}^{i,\,gf}_{l,\,ba}\Big|_{\text{div}} + i\,\not{p}\,\delta Z_{l,\,gf}\,P_L\,\delta_{ba} \tag{II-2.208}$$

for the lepton-doublet, and



(II-2.209)



(II-2.210)

$$= \sum_i \mathcal{B}^{i}_{\phi,\,ba}\Big|_{\text{div}} + i\,p^2\,\delta Z_{\phi}\,\delta_{ba} \tag{II-2.211}$$

for the Higgs-doublet. Note that we have defined the divergent part such that it still includes the $1/(4-d)$ pole. Therefore, we find for the $\delta Z_{l/\phi,\,1}$ without the $1/(4-d)$ poles:

$$\delta Z_{l,\,gf,\,1} = -\frac{4-d}{i\,\not{p}\,P_L\,\delta_{ba}} \sum_i \mathcal{B}^{i,\,gf}_{l,\,ba}\Big|_{\text{div}}, \tag{II-2.212}$$

$$\delta Z_{\phi,\,1} = -\frac{4-d}{i\,p^2\,\delta_{ba}} \sum_i \mathcal{B}^{i}_{\phi,\,ba}\Big|_{\text{div}}, \tag{II-2.213}$$

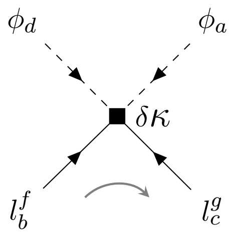
where the fraction of tensor-valued quantities is understood as dropping them in the corresponding expressions for the divergent diagrams, and we do not sum over repeated indices. Note that the factor of $(4-d)$ cancels the poles from UV-divergences in the 1PI diagrams, such that $\delta Z_{l/\phi,\,1}$ are independent of d.

While this procedure is, fundamentally, the usual paradigm, we present the formulae in a more convenient and directly usable form possible here to facilitate calculations and avoid, e.g., sign errors. In particular, the equations we present here include our previous considerations concerning the fact that only bubble diagrams contribute, which will simplify future calculations. Note that the direct usability and ease of implementation is also the reason behind defining $\mathcal{B}^i$ as representing the entire respective diagram, and not defining it as $i\,\mathcal{B}^i$, as is usually the case.

Let us now move on to the vertex corrections. We recall that the relevant counterterm has the form

$$\mathcal{L}_{tot} \supset \mathcal{C}_{\mathcal{K}} = \frac{1}{4}\,\delta\mathcal{K}_{gf}\,\overline{l_c^{g,C}}\,\varepsilon^{cd}\,\phi_d\,l_b^f\,\varepsilon^{ba}\,\phi_a\;+\;\text{h.c.}\,, \tag{II-2.214}$$

which leads to the Feynman-rule



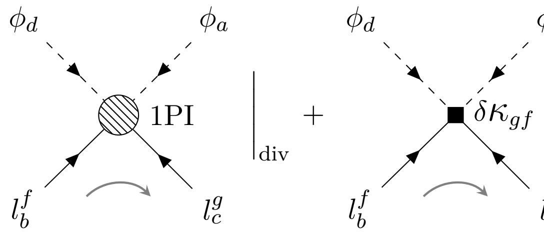
$$= i\,\tilde{\mu}^{(4-d)}\delta\mathcal{K}_{gf}\;\underbrace{\frac{1}{2}\big(\varepsilon_{cd}\,\varepsilon_{ba}+\varepsilon_{ca}\,\varepsilon_{bd}\big)}_{\equiv E_{abcd}}\;P_L\;=\;i\,\tilde{\mu}^{(4-d)}\,\delta\mathcal{K}_{gf}\,E_{abcd}\,P_L\,. \tag{II-2.215}$$

Where we introduced the dimensional rescaling factor $\tilde{\mu}^{D_{\mathcal{K}}(4-d)} \overset{D_{\mathcal{K}}=1}{=} \tilde{\mu}^{(4-d)}$ in the $\overline{\text{MS}}$ scheme for the Feynman-rules for $\delta\mathcal{K}$—and thus $\mathcal{K}$—here. The scaling factor is given by

$$\tilde{\mu} \equiv \mu\,\sqrt{\frac{e^{\gamma_E}}{4\pi}}\,. \tag{II-2.216}$$

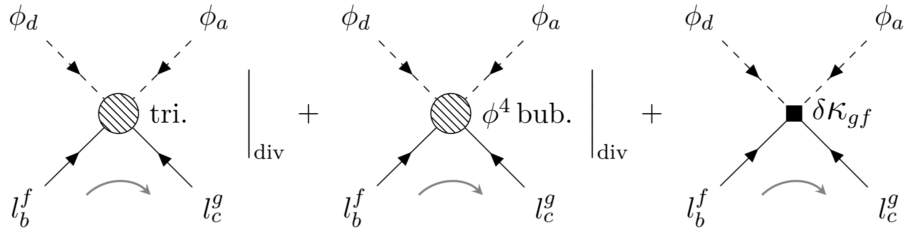
The counterterm needs to cancel the UV-divergences in vertex correction diagrams. As before, this means that the sum of divergent parts of the relevant 1PI diagrams and the counterterm has to give zero. Once again, we can make use of our previous knowledge from earlier sections; we know that apart from the quartic Higgs self-coupling, only triangle diagrams containing $\mathcal{K}$ can contribute at the one-loop level. Thus, we only need to consider these types of diagrams, which we will denote as $\mathcal{T}_{\mathcal{K}}$ for the triangles, and $\mathcal{B}_{\phi^4}$ for the quartic Higgs bubble. Diagrammatically, this gives us

$$0 = \sum_{\substack{\text{1PI} \\ \text{diagrams}}}$$



(II-2.217)

$$= \sum_{\text{triangles}}$$



(II-2.218)

$$= \sum_i \mathcal{T}^{i,\,gf}_{\mathcal{K},\,abcd}\Big|_{\text{div}} + \mathcal{B}_{\phi^4} + i\,\tilde{\mu}^{(4-d)}\,\delta\mathcal{K}_{gf}\,E_{abcd}\,P_L\,. \tag{II-2.219}$$

Inverting this relation, we find for $\delta\mathcal{K}_{,1}$:

$$\delta\mathcal{K}_{gf,1} = -\frac{4-d}{i\,\tilde{\mu}^{(4-d)}\,E_{abcd}\,P_L}\left[\sum_i \mathcal{T}^{i,\,gf}_{\mathcal{K},\,abcd}\Big|_{\text{div}} + \mathcal{B}^{gf}_{\phi^4,\,abcd}\Big|_{\text{div}}\right], \tag{II-2.220}$$

where we do not sum over repeated indices.

We note that we have written $(4-d)$ explicitly, instead of rewriting this in terms of four and ε dimensions to facilitate calculating the diagrams themselves in either $d = 4-\varepsilon$ or $d = 4-2\varepsilon$, while avoiding possible mismatch between the resulting renormalization constants. The reason for this is the following. If we choose $d = 4-\varepsilon$, the corresponding $\beta_{\mathcal{K}}$-function is defined as

$$\beta_{\mathcal{K}} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \mu\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\mu}\,, \tag{II-2.221}$$

whereas choosing $d = 4-2\varepsilon$ yields the $\beta_{\mathcal{K}}$-function

$$\beta_{\mathcal{K}} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \mu^2\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}\mu^2}\,. \tag{II-2.222}$$

In principle, these are the same; however, we note that the μ of eq. (II-2.221) has mass dimension 2, while the μ of eq. (II-2.222) has mass dimension 1. Furthermore, rewriting, e.g., eq. (II-2.222) with a logarithmic derivative of μ to the power of 1 would lead to an additional factor of 1/2:

$$\mu^2\,\frac{\mathrm{d}}{\mathrm{d}\mu^2} = \mu^2\,\frac{\mathrm{d}}{2\mu\cdot\mathrm{d}\mu} = \frac{1}{2}\cdot\mu\,\frac{\mathrm{d}}{\mathrm{d}\mu}\,, \tag{II-2.223}$$

which can lead to confusion and mismatch when comparing form-equivalent results computed in different parametrizations of d. In terms of the renormalization constants, this arises in the following way. For brevity, let us name the ε in $d = 4 - 2\varepsilon$ dimensions ε_2, and the one in $d = 4 - \varepsilon$ dimensions ε_1. If we now consider, for example, a field renormalization result computed in $d = 4 - 2\varepsilon_2$ dimensions, and only factor out the $1/\varepsilon_2$ instead of the full $(4 - d)$, we obtain

$$\delta Z_{\varepsilon_2,1} \stackrel{!}{=} \varepsilon_2 \cdot \delta Z_{\varepsilon_2} \equiv \varepsilon_2 \cdot z \frac{1}{\varepsilon_2} \tag{II-2.224}$$

$$= z \,, \tag{II-2.225}$$

whereas $d = 4 - \varepsilon_1$ yields

$$\delta Z_{\varepsilon_1,1} \stackrel{!}{=} \varepsilon_1 \cdot \delta Z_{\varepsilon_1} \equiv \varepsilon_1 \cdot z \frac{1}{\varepsilon_1} \tag{II-2.226}$$

$$= z \,. \tag{II-2.227}$$

However, there is a mismatch between these two! Let us see where this mismatch comes from:

$$\delta Z_{\varepsilon_2} = z_{\varepsilon_2} \frac{1}{\varepsilon_2} = z_{\varepsilon_2} \frac{2}{2 \cdot \varepsilon_2} \tag{II-2.228}$$

$$\stackrel{\varepsilon_1 = 2\varepsilon_2}{=} \underbrace{2 z_{\varepsilon_2}}_{= z_{\varepsilon_1}\ !} \frac{1}{\varepsilon_1} \tag{II-2.229}$$

So we see that the mismatch comes from erroneously identifying $z_{\epsilon_2} = z_{\varepsilon_1}$, which comes from factoring out ε in a form-equivalent way. Therefore, to get the correct results, we need to consistently factor out $(4 - d)$, and not just ε itself.

In writing eq. (II-2.212), (II-2.213), and (II-2.220) with an explicit factor of $(d - 4)$ we can avoid such mismatches, as the correct factors of 2, 1/2, etc. are automatically introduced. Furthermore, this allows us to calculate the diagrams in whichever parametrization of d we find most convenient, and nevertheless obtain results that can be directly compared to other works. In this work, for instance, we will calculate the diagrams in $d = 4 - 2\varepsilon_2$ dimensions, but write the $\beta_{\mathcal{K}}$-function in terms of the $d = 4 - \varepsilon_1$ parametrization $\beta_{\mathcal{K}} = \mu_{\varepsilon_1}\, \mathrm{d}\mathcal{K}/\mathrm{d}\mu_{\varepsilon_1}$, as this is more widely used in this context. Therefore, a correct identification of the $\delta Z_{l/\phi,1}$ and $\delta\mathcal{K}_{,1}$ is crucial. Note also that in writing $\beta_{\mathcal{K}} = \mathrm{d}\mathcal{K}/\mathrm{d}t$, we can avoid possible confusion regarding the dimensionality of μ—although, to be precise, the definition of t depends on whether we use μ_{ε_1} or μ_{ε_2}, via $t = \ln \mu_{\varepsilon_1} = \ln \mu_{\varepsilon_2}^2$. We will henceforth drop the indices on ε to indicate the parametrization of d it originates from.

Let us now summarize the main points discussed in this subsection.

When **calculating the renormalization constants from diagrams**, we can make use of our previous knowledge of the contributig topologies, simplifying our considerations. We know that for the Weinberg-Operator, **only bubbles contribute to the wave function renormalization, and only triangles and the quartic Higgs bubble contribute to the vertex renormalization at the one-loop level.** Thus, the renormalization constants are given by

$$\delta Z_{l,\,gf,\,1} = -\frac{4-d}{i\,\not{p}\,P_L\,\delta_{ba}}\,\sum_i \mathcal{B}^{i,\,gf}_{l,\,ba}\Big|_{\text{div}}\,, \tag{II-2.230}$$

$$\delta Z_{\phi,\,1} = -\frac{4-d}{i\,p^2\,\delta_{ba}}\,\sum_i \mathcal{B}^{i}_{\phi,\,ba}\Big|_{\text{div}}\,, \qquad \text{and} \tag{II-2.231}$$

$$\delta \mathcal{K}_{gf,\,1} = -\frac{4-d}{i\,\tilde{\mu}^{(4-d)}\,E_{abcd}\,P_L}\left[\sum_i \mathcal{T}^{i,\,gf}_{\mathcal{K},\,abcd}\Big|_{\text{div}} + \mathcal{B}^{gf}_{\phi^4,\,abcd}\Big|_{\text{div}}\right]. \tag{II-2.232}$$

We define the **calligraphic quantities** as **entire diagrams**. $\mathcal{B}^i_{l/\phi}$ are self-energy bubbles of the lepton- and Higgs-doublet, $\mathcal{B}_{\phi^4}$ is the bubble from the vertex correction via quartic Higgs self-coupling, and $\mathcal{T}^i_{\mathcal{K}}$ are vertex correction triangles. Note that we do not explicitly include the dimensional rescaling factor for $\delta\mathcal{K}$. The denominators with tensor-valued quantities are understood as dropping them in the corresponding expressions for the divergent diagrams, and we do not sum over repeated indices. We also define the short-hand notation $\boldsymbol{E_{abcd} = \frac{1}{2}\left(\varepsilon_{cd}\,\varepsilon_{ba} + \varepsilon_{ca}\,\varepsilon_{bd}\right)}$. Furthermore, we define the **divergent part including the poles in $\boldsymbol{(4-d)}$**. Thus, diagrams calculated in $d = 4 - 2\varepsilon$ dimensions can also be consistently implemented in $\beta_{\mathcal{K}}$ in $d = 4 - \varepsilon$ dimensions.

II-2.3.3 Loop-Integral, and Coupling Structure Universality of the New Quantum Effect

Let us now focus on the new terms discovered in this work, and calculate them in full generality. We recall, these are terms of the form

$$\beta_\kappa \supset \sum_r \left(G^{(r)}\right)^T \kappa\, G^{(r)} + \frac{1}{2}\sum_s \left[\left(G_+^{(s)}\right)^T \kappa\, G_-^{(s)} + \left(G_-^{(s)}\right)^T \kappa\, G_+^{(s)}\right]. \qquad \text{(II-2.233)}$$

They arise from triangle diagrams with flavor-nonuniversally coupling vector bosons being exchanged between the lepton-doublet legs; the $G^{(r)}$ come from neutral vectors, and the $G_\pm^{(s)}$ from flavor-charged ones. The corresponding diagrams are of the form as displayed in fig. (II-2.10), which result in terms with $G \sim T$ and $G_\pm \sim T_\pm$.

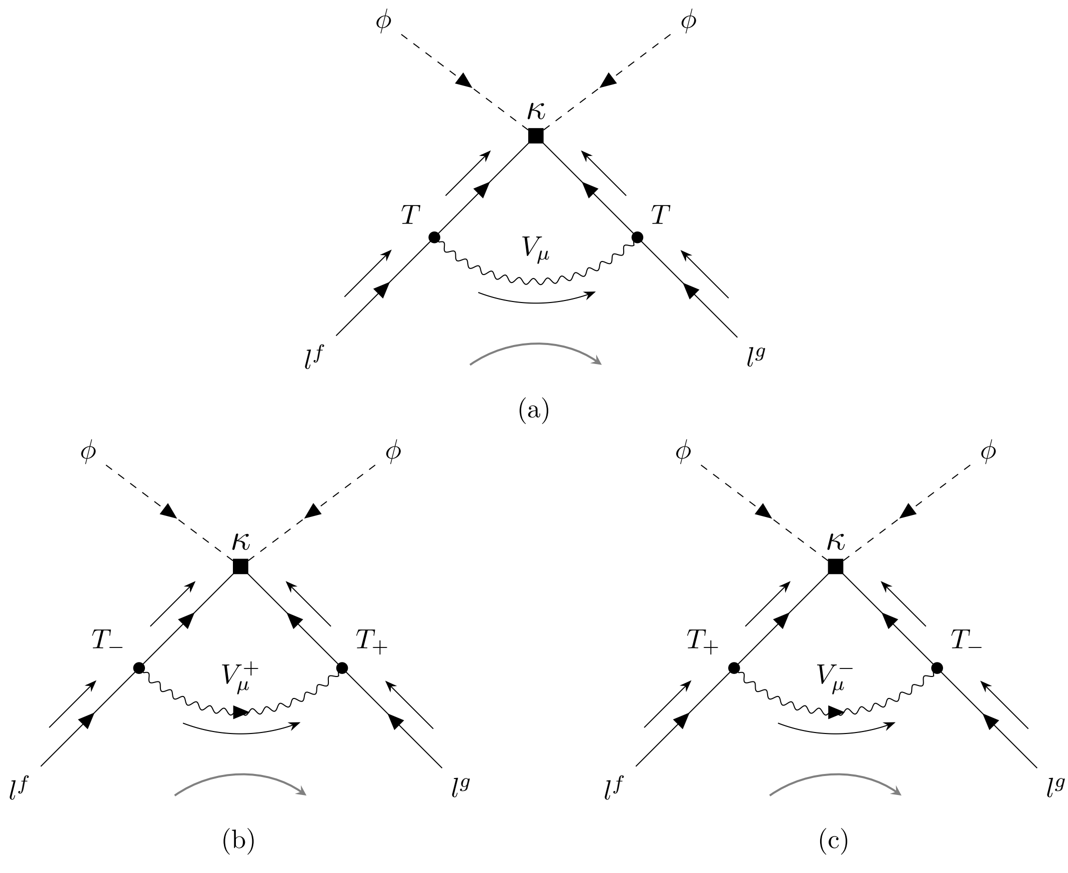


Figure II-2.10.: Contribution of some neutral vector boson V_μ, and charged vector bosons $V_\mu^\pm$ to the Weinberg-Operator via vertex corrections; the neutral vector contribution is displayed in fig. (II-2.10a), the positive, in fig. (II-2.10b), and the negative, in fig. (II-2.10c); T and $T_\pm$ are matrices in flavor-space coming from the couplings between the respective vector bosons and the lepton-doublets; the superscript is the lepton-family index; the gray arrows denote fermion flow

It turns out that we can calculate the contribution coming from these terms in general, such that they may be used in future works, and simplify the application to specific models. The latter will be particularly useful in the next chapter, when discussing specific models of flavor gauge extensions. To set up this calculation, we define the interactions

$$\mathcal{L} \supset -g_n\, T_{gf}\, \overline{l^g}\, \gamma^\mu\, l^f\, V_\mu - \frac{1}{\sqrt{2}}\left[g_c\, T_{+,\,gf}\, \overline{l^g}\, \gamma^\mu\, l^f\, V_\mu^+ + \text{h.c.}\right], \tag{II-2.234}$$

where we have defined the couplings g_n and g_c to neutral and charged vector bosons V_μ and $V_\mu^\pm$, respectively. Furthermore, we defined the corresponding coupling matrices in flavor-space T and $T_\pm$ with $T_\pm^\dagger = T_\mp$. Note that we can always freely choose the normalization, such as $1/\sqrt{2}$, by rescaling g_c—in this case we choose it like this to make the relation with the redefinition from neutral to charged basis manifest.

These interactions are written in exactly the form they would emerge from gauge covariant derivatives,

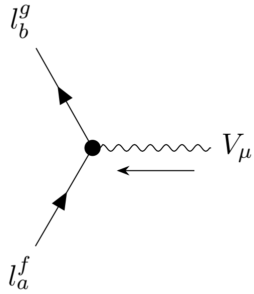
$$\overline{l^g}\, i\, \gamma^\mu\, D_\mu^{gf}\, l^f = \overline{l^g}\, i\, \gamma^\mu \left(\delta_{gf}\, \partial_\mu + i\, g_n\, V_\mu\, T_{gf} + \frac{i}{\sqrt{2}}\, g_c\, V_\mu^+\, T_{+,\,gf} + \frac{i}{\sqrt{2}}\, g_n\, V_\mu^-\, T_{-,\,gf}\right) l^f \tag{II-2.235}$$

$$= \overline{l^g}\, i\, \delta_{gf} \gamma^\mu\, \partial_\mu\, l^f - g_n\, T_{gf}\, \overline{l^g}\, \gamma^\mu\, l^f\, V_\mu + \tag{II-2.236}$$

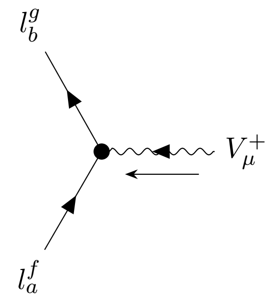
$$- \frac{1}{\sqrt{2}}\left[g_c\, T_{+,\,gf}\, \overline{l^g}\, \gamma^\mu\, l^f\, V_\mu^+ + \text{h.c.}\right].$$

Note that for the independent, massive, vector boson, we may write the interaction in this way as well. Furthermore, the neutral and charged bosons are not necessarily related, they may be but are not required to. We employ this notation here to calculate the diagrams for a general neutral vector boson, and two coupled charged ones; we can then generalize this by summing over multiple such contributions, and if desired relate them.

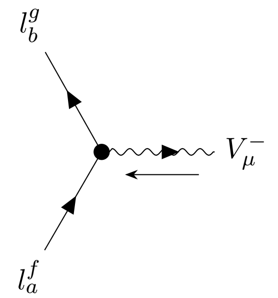
The Feynman-rules obtained from the interactions (II-2.234) are



$$= -i\, \tilde{\mu}^{(4-d)/2}\, g_n\, T_{gf}\, \gamma_\mu\, P_L\, \delta_{ba}\,, \tag{II-2.237}$$



$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T_{+,\,gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{II-2.238}$$



$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T_{-,\,gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{II-2.239}$$

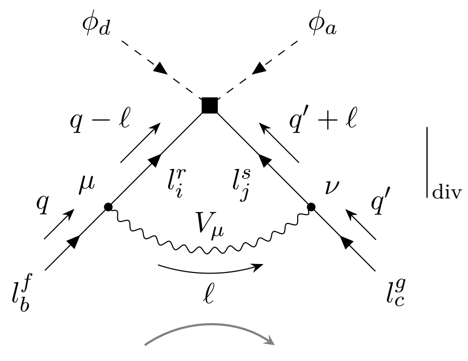
where we have used that $D_V = D_3 = \frac{1}{2}$.

With this, we can now calculate the diagrams of fig. (II-2.10). Note that we perform the d-dimensional integral in the $\overline{\text{MS}}$ scheme with $d = 4 - 2\varepsilon$ and define the integral measure including a dimensional scaling factor from the Feynman-rules as

$$\int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \quad \longrightarrow \quad \tilde{\mu}^{(4-d)} \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\,, \qquad \text{with} \quad \tilde{\mu} = \mu\sqrt{\frac{e^{\gamma_E}}{4\pi}} \tag{II-2.240}$$

$$\xrightarrow{d=4-2\varepsilon} \quad \tilde{\mu}^{2\varepsilon} \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \;\equiv\; \int \left[\mathrm{d}^d\ell\right] \tag{II-2.241}$$

For the neutral vector, we thus obtain



$$\Big|_{\text{div}} \quad = \tag{II-2.242}$$

$$= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\left[-\,i\,\tilde{\mu}^{\varepsilon}\,g_n\,\boldsymbol{T_{sg}}\left(-\,\gamma_\nu\,P_R\right)\delta_{jc}\right]\frac{i\left(-\,\not{q}'-\not{\ell}\right)}{(q'+\ell)^2}\left[i\,\tilde{\mu}^{2\varepsilon}\boldsymbol{\kappa_{sr}}\,E_{aijd}\,P_L\right] \quad \times \tag{II-2.243}$$

$$\times \quad \frac{i\,(\not{q}-\not{\ell})}{(q-\ell)^2}\Big[-i\,\tilde{\mu}^{\varepsilon}\,g_n\,\boldsymbol{T_{rf}}\,\gamma_\mu\,P_L\,\delta_{ib}\Big]\frac{i}{\ell^2-M_V}\left[-g^{\mu\nu}+(1-\xi_n)\,\frac{\ell^\mu\,\ell^\nu}{\ell^2-\xi_n\,M_V^2}\right]$$

$$= (-i\,g_n)^2\,\tilde{\mu}^{2\varepsilon}\,\big(\boldsymbol{T^T}\,\boldsymbol{\kappa}\,\boldsymbol{T}\big)_{\boldsymbol{gf}}\,E_{abcd}\,P_L\cdot\int\big[\mathrm{d}^d\ell\big]\,(-\gamma_\nu)\,\frac{(-\not{q}'-\not{\ell})}{(q'+\ell)^2}\,\frac{(\not{q}-\not{\ell})}{(q-\ell)^2}\,\gamma_\mu \quad\times \tag{II-2.244}$$

$$\times \quad \frac{1}{\ell^2-M_V}\left[-g^{\mu\nu}+(1-\xi_n)\,\frac{\ell^\mu\,\ell^\nu}{\ell^2-\xi_n\,M_V^2}\right]$$

$$= (-i\,g_n)^2\,\tilde{\mu}^{2\varepsilon}\,\big(\boldsymbol{T^T}\,\boldsymbol{\kappa}\,\boldsymbol{T}\big)_{\boldsymbol{gf}}\,E_{abcd}\,P_L\cdot\frac{i}{16\pi^2}\,(3+\xi_n)\,\frac{1}{\varepsilon} \tag{II-2.245}$$

$$= -\frac{i}{16\pi^2}\,\tilde{\mu}^{2\varepsilon}\,E_{abcd}\,P_L\,\frac{1}{\varepsilon}\,g_n^2\,(3+\xi_n)\,\big(\boldsymbol{T^T}\,\boldsymbol{\kappa}\,\boldsymbol{T}\big)_{\boldsymbol{gf}}\,. \tag{II-2.246}$$

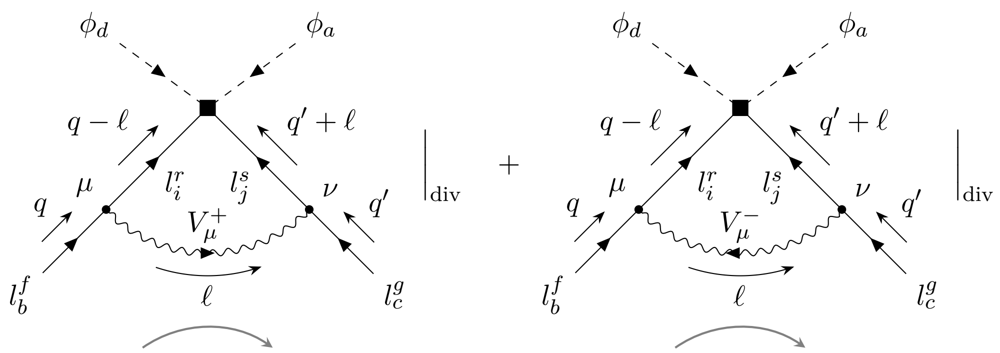
We have made the flavor-space couplings bold in the calculation to emphasize the emerging couplings structure. Furthermore, we included a mass for the vector bosons for generality, but as expected from previous discussions, the result is independent of it. We can now directly employ eq. (II-2.232) to obtain the corresponding contribution to $\delta\boldsymbol{\kappa}$, referred to by llV_μ:

$$\delta\kappa_{gf,1}\Big|_{llV_\mu} = -\frac{2\varepsilon}{i\,\tilde{\mu}^{2\varepsilon}\,E_{abcd}\,P_L}\cdot\mathcal{T}^{llV_\mu,\,gf}_{\mathcal{K},\,abcd}\Big|_{\mathrm{div}} \tag{II-2.247}$$

$$= -\frac{2\varepsilon}{i\,\tilde{\mu}^{2\varepsilon}\,E_{abcd}\,P_L}\cdot\Big[-\frac{i}{16\pi^2}\,\tilde{\mu}^{2\varepsilon}\,E_{abcd}\,P_L\,\frac{1}{\varepsilon}\,g_n^2\,(3+\xi_n)\,\big(\boldsymbol{T^T}\,\boldsymbol{\kappa}\,\boldsymbol{T}\big)_{\boldsymbol{gf}}\Big] \tag{II-2.248}$$

$$= \frac{2}{16\pi^2}\,g_n^2\,(3+\xi_n)\,\big(\boldsymbol{T^T}\,\boldsymbol{\kappa}\,\boldsymbol{T}\big)_{\boldsymbol{gf}}\,, \tag{II-2.249}$$

where we can directly see the benefit of the notation in eq. (II-2.230)–(II-2.232) in terms of usability. Before discussing this result, let us also calculate the contribution coming from charged vector bosons $V_\mu^\pm$:



= (II-2.250)

$$= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \Big[-\frac{i}{\sqrt{2}}\,\tilde{\mu}^{\varepsilon}\, g_c\, \boldsymbol{T_{+,\,sg}} \big(-\gamma_\nu\, P_R \big)\, \delta_{jc} \Big]\, \frac{i\, \big(-\not{q}' - \not{\ell} \big)}{(q'+\ell)^2} \quad \times \tag{II-2.251}$$

$$\times \quad \Big[i\, \tilde{\mu}^{2\varepsilon} \boldsymbol{\kappa_{sr}}\, E_{aijd}\, P_L \Big]\, \frac{i\, (\not{q} - \not{\ell})}{(q-\ell)^2} \Big[-\frac{i}{\sqrt{2}}\,\tilde{\mu}^{\varepsilon}\, g_c\, \boldsymbol{T_{-,\,rf}}\, \gamma_\mu\, P_L\, \delta_{ib} \Big]\, \frac{i}{\ell^2 - M_V} \quad \times$$

$$\times \quad \Big[-g^{\mu\nu} + (1-\xi_c)\, \frac{\ell^\mu\, \ell^\nu}{\ell^2 - \xi_c\, M_V^2} \Big] \quad + \quad (+ \longleftrightarrow -)$$

$$= (-i\, g_c)^2\, \tilde{\mu}^{2\varepsilon}\, \boldsymbol{\frac{1}{2}} \Big(\boldsymbol{T_+^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_-} + \boldsymbol{T_-^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_+} \Big)_{\boldsymbol{gf}}\, E_{abcd}\, P_L \cdot \int \big[\mathrm{d}^d\ell \big]\, \big(-\gamma_\nu \big) \quad \times \tag{II-2.252}$$

$$\times \quad \frac{\big(-\not{q}' - \not{\ell} \big)}{(q'+\ell)^2}\, \frac{\big(\not{q} - \not{\ell} \big)}{(q-\ell)^2}\, \gamma_\mu\, \frac{1}{\ell^2 - M_V} \Big[-g^{\mu\nu} + (1-\xi_c)\, \frac{\ell^\mu\, \ell^\nu}{\ell^2 - \xi_c\, M_V^2} \Big]$$

$$= (-i\, g_c)^2\, \tilde{\mu}^{2\varepsilon}\, \boldsymbol{\frac{1}{2}} \Big(\boldsymbol{T_+^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_-} + \boldsymbol{T_-^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_+} \Big)_{\boldsymbol{gf}}\, E_{abcd}\, P_L \cdot \frac{i}{16\pi^2}\, g_c^2 (3+\xi_c)\, \frac{1}{\varepsilon} \tag{II-2.253}$$

$$= -\frac{i}{16\pi^2}\, \tilde{\mu}^{2\varepsilon}\, E_{abcd}\, P_L\, \frac{1}{\varepsilon}\, g_c^2\, (3+\xi_c)\; \boldsymbol{\frac{1}{2}} \Big(\boldsymbol{T_+^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_-} + \boldsymbol{T_-^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_+} \Big)_{\boldsymbol{gf}}\,. \tag{II-2.254}$$

Employing eq. (II-2.232) again, we obtain the contribution of the charged vectors,

$$\delta\mathcal{K}_{gf,\,1}\, \Big|_{llV_\mu^\pm} = -\frac{2\varepsilon}{i\, \tilde{\mu}^{2\varepsilon}\, E_{abcd}\, P_L} \cdot \mathcal{T}^{llV_\mu^\pm,\,gf}_{\mathcal{K},\,abcd}\, \Big|_{\mathrm{div}} \tag{II-2.255}$$

$$= -\frac{2\varepsilon}{i\, \tilde{\mu}^{2\varepsilon}\, E_{abcd}\, P_L} \cdot \Big[-\frac{i}{16\pi^2}\, \tilde{\mu}^{2\varepsilon}\, E_{abcd}\, P_L\, \frac{1}{\varepsilon}\, g_c^2\, (3+\xi_c) \quad \times \tag{II-2.256}$$

$$\times \quad \boldsymbol{\frac{1}{2}} \Big(\boldsymbol{T_+^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_-} + \boldsymbol{T_-^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_+} \Big)_{\boldsymbol{gf}} \Big]$$

$$= \frac{2}{16\pi^2}\, g_c^2\, (3+\xi_c)\; \boldsymbol{\frac{1}{2}} \Big(\boldsymbol{T_+^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_-} + \boldsymbol{T_-^T}\, \boldsymbol{\kappa}\, \boldsymbol{T_+} \Big)_{\boldsymbol{gf}}\,. \tag{II-2.257}$$

We note that we have performed the loop-integrals both by hand, and using the Mathematica Package FeynCalc [44, 45, 46]. To calculate the integrals manually, for instance, we reduce tensor integrals to scalar integrals by rewriting scalar products of the loop momentum in terms of linear combinations of propagators, and introduce Feynman-parameters to reduce the various resulting scalar integrals to tadpoles with a higher-power propagator. These methods are, e.g., described in [47]. To calculate the integrals in FeynCalc, we use the built-in tensor integral reduction to Passarino-Veltman functions and their known UV divergences—Passarino-Veltman functions provide a parametrization of one-loop tensor integrals [48, 49].

Let us also restate that we followed [38] to obtain the correct Feynman-rules using fermion flow in fermion-number-violating interactions.

Let us now discuss these results. We notice multiple things, such as that

① the contribution arising from the neutral vector boson is exactly of the expected *positive* $G^T\,\mathcal{K}\,G$ form,

② the contribution arising from the charged vector bosons is exactly of the expected *positive* $G_+^T\,\mathcal{K}\,G_- + G_-^T\,\mathcal{K}\,G_+$ form;

③ the contributions are *independent of the masses of the vector bosons*;

④ the contributions are gauge-dependent;

⑤ the contributions arise through the same loop-integral;

⑥ the contributions only differ in their gauge parameter, coupling constant and flavor-space coupling matrices; and

⑦ the contributions are valid for *any number of lepton generations, and potential gauge groups and representations.*

Let us now discuss these points in detail.

① and ② : There are two points in these statements. First, the contributions from vertex corrections via flavor-nonuniversally coupling, neutral and charged vector bosons exchanged between the lepton-doublets are exactly of the form as claimed. This is perhaps not surprising, given the extensive discussion of possible topologies that can give rise to $\sim G$, and $\sim G_\pm$ terms. However, we notice that indeed, following eq. (II-2.232), the contributions from different vectors that are not in conjugate $V_\mu^\pm$ pairs do not mix with each other. Their corrections are simply summed in $\delta\mathcal{K}$, consistent with eq. (II-2.21) derived at the beginning of this chapter. Second, the sign of the contributions is *positive*, as we had claimed earlier. This justifies the usage of only positive coefficients in eq. (II-2.21). Third, we see that the relations $G \sim T$ and $G_\pm \sim T_\pm$ hold in direct correspondence to one another, i.e.,

$$G = \text{const.}_n \cdot T\,, \qquad G_\pm = \text{const.}_c \cdot T_\pm\,, \tag{II-2.258}$$

where the prefactors const._n and const._c are completely form-identical. We only replace $g_n,\, \xi_n \longleftrightarrow g_c\,, \xi_c$; and in particular, if $g_n = g_c$ and $\xi_n = \xi_c$—which is the case if these come from the same gauge group—the prefactors are *completely identical.*

③ : We see explicitly that even if we put a non-zero mass for the vector bosons, the resulting contributions to the $\beta_{\mathcal{K}}$-function *are indeed independent of the masses.* We had made this point previously using power-counting arguments and two-point insertions of mass-terms, but this provides an explicit proof that the arguments hold true.

(4) : The results of eq. (II-2.249) and (II-2.257) are gauge-dependent, since the gauge parameters ξ_n and ξ_c appear explicitly. In principle, this is not a problem; individual diagrams may be gauge-dependent. Generally, only *physical* quantities need to be gauge-invariant, and individual diagrams or sums of such are not physical—recall that from an experimental standpoint different diagrams for the same external states are indistinguishable. Therefore, the quantity that needs to be gauge-independent is the total *physical* $\beta_{\mathcal{K}}$-function, and the individual diagrams being gauge-dependent is not a problem—as long as the gauge-dependence cancels in the end. We will comment more on issues with gauge-invariance later.

(5) and (6) : We observe that the integrals and resulting contributions are *universal*. First, we can parametrize the loop-integral itself such that it has the exact same structure for both the neutral and charged vectors, and independently of the specific couplings; this can be seen in eq. (II-2.244) and (II-2.252). Furthermore, the resulting flavor-space structure and numeric prefactor are such that for any type of coupling, it is only necessary to perform substitutions between the coupling constants, matrices, and gauge parameters. Thereby, *any* contribution stemming from this loop-topology is given by the formulae in eq. (II-2.249) and (II-2.257), *including flavor-universal couplings.*

Let us consider a concrete example to illustrate these points. For instance, the Standard Model result can be readily obtained by substituting in eq. (II-2.249)

- $\xi_n \longrightarrow \xi_B$, $g_n \longrightarrow g_1 = g'$, and $T \longrightarrow q_Y \mathbb{1} = -\frac{1}{2}\mathbb{1}$ for $U(1)_Y$; and
- $\xi_n \longrightarrow \xi_W$, $g_n \longrightarrow g_2 = g$, and $T^i \longrightarrow \frac{\tau^i}{2} \otimes \mathbb{1}$ for $SU(2)_L$.

In the $U(1)_Y$ case, we substitute by the unit matrix in flavor-space, and multiply with the hypercharge of the lepton doublet, giving exactly the expected SM result of [34],

$$\delta\mathcal{K}_{gf,1}\Big|_{llV_\mu = llB_\mu} = \frac{2}{16\pi^2}\, g_n^2\,(3+\xi_n)\,\left(T^T\,\mathcal{K}\,T\right)_{gf}\Big|_{llV_\mu = llB_\mu} \tag{II-2.259}$$

$$\longrightarrow \frac{2}{16\pi^2}\, g_1^2\,(3+\xi_B)\,\frac{1}{4}\,\mathcal{K}_{gf} = \frac{1}{32\pi^2}\, g_1^2\,(3+\xi_B)\,\mathcal{K}_{gf} \quad ✓\,. \tag{II-2.260}$$

In the $SU(2)_L$ case, we need to keep in mind that the coupling matrices to the lepton doublets may be $\sim \mathbb{1}$ in flavor-space, *but not in $SU(2)_L$-space*! This therefore provides a great example of how to handle such situations. Namely, we substitute the three generators by 1/2 times the Pauli matrices in $SU(2)_L$-space, and take a symbolic tensor product with the unit matrix in flavor-space. Then, given that we have contributions from all three gauge bosons W^1, W^2, and W^3, we need to sum over all their contributions. Then, we recall that the lepton-doublets are in the fundamental representation of $SU(2)_L$, so we can make use of eq. (II-2.70) that we derived in the previous section,

$$\sum_A \left(T^A\right)^T T^A = \frac{N-1}{2N}\,\mathbb{1}, \tag{II-2.261}$$

which, in our case, corresponds to non-flavor spaces. Recall that we derived this using the completeness relation for $SU(N)$ generators in the fundamental representation. For $SU(2)$ we obtain

$$\sum_{i=1}^{3} \left(\frac{\tau^i}{2}\right)^T \frac{\tau^i}{2} = \frac{2-1}{2\cdot 2}\,\mathbb{1} = \frac{1}{4}\,\mathbb{1}\,, \tag{II-2.262}$$

and thus

$$\delta\kappa_{gf,1}\Big|_{llV_\mu = llW_\mu} = \frac{2}{16\pi^2}\, g_n^2\,(3+\xi_n) \sum_{i=1}^{3} \left[\left(T^i\right)^T \kappa\, T\right]_{gf} \tag{II-2.263}$$

$$\longrightarrow \frac{2}{16\pi^2}\, g_2^2\,(3+\xi_W)\,\frac{1}{4}\,\kappa_{gf} = \frac{1}{32\pi^2}\, g_2^2\,(3+\xi_W)\,\kappa_{gf} \quad ✓\,, \tag{II-2.264}$$

which is consistent with previous SM results [34].

(7) : We notice that at no point did we assume any particular dimensionality of κ or the couplings in flavor-space. This means that the derived contributions are valid for *any number of lepton generations*. While we only have three generations in the Standard Model, we can imagine models with more; for instance, sterile neutrinos. Sterile neutrinos provide a possible explanation of neutrino masses, and can be regarded as additional flavors [50]. While this is just an example, it makes it apparent that results that hold for any number of lepton generations are attractive. In this context, this means that the presented results can be applied to a wide range of theories, including those with extended lepton flavor sectors. Furthermore—for flavor gauge theories—we did not assume any particular gauge group or representation of the leptons under this gauge group. This means that we can apply the results not only for any number of lepton generations, but for any reasonable group and representation that can accommodate them. For instance, if we assume three lepton generations and wanted to build a model that includes a flavor gauge theory, these results hold for any gauge group with fitting representations. In particular, that includes any charge assignment of $U(1)$, a doublet + singlet representation of $SU(2)$, a triplet representation of $SU(2)$, and the fundamental representation of $SU(3)$. Incidentally, we will explore and build explicit models for $U(1)$, $SU(2)$ triplet, and $SU(3)$ fundamental theories in the next chapter.

Let us now go back to the results of eq. (II-2.249) and apply it to an $SU(N)$ flavor gauge theory. In particular, let us consider the case where the leptons are in the fundamental representation. Once again, the fundamental representation exhibits special simplifications by virtue of the completeness relation, eq. (II-2.62).

We take the flavor coupling structure of eq. (II-2.249), $T^T\,\kappa\,T$, and now sum over all $SU(N)$ generators. This corresponds to summing all diagrams with intermediate gauge bosons in this $SU(N)$—recall also that the result is independent of which basis we work with, so we can choose the neutral basis and apply the completeness relation. We thus obtain for the jl-component of the coupling structure,

$$\sum_{A=1}^{N^2-1}\Big[\big(T^A\big)^T\,\kappa\,T^A\Big]_{jl} = \sum_{A=1}^{N^2-1} T^A_{ij}\,\kappa_{ik}\,T^A_{kl} \tag{II-2.265}$$

$$= \kappa_{ik}\sum_{A=1}^{N^2-1} T^A_{ij}\,T^A_{kl} \tag{II-2.266}$$

$$\overset{(II-2.62)}{=} \kappa_{ik}\,\frac{1}{2}\Big(\delta_{il}\,\delta_{jk} - \frac{1}{N}\,\delta_{ij}\,\delta_{kl}\Big) \tag{II-2.267}$$

$$= \frac{1}{2}\big(\kappa_{lj} - \frac{1}{N}\,\kappa_{jl}\big) \tag{II-2.268}$$

$$\overset{\kappa^T=\kappa}{=} \frac{N-1}{2N}\,\kappa_{jl}\,. \tag{II-2.269}$$

Indeed, we could have also obtained this from eq. (II-2.63) by substituting $\delta_{jl} \longrightarrow \kappa_{jl}$, however, this provides a more direct derivation.

What this result means is that in the fundamental representation of $SU(N)$ flavor gauge theories, the $llSU(N)$ vertex corrections *do not lead to* $\sim G_{(\pm)}$ *terms, but contribute only to the* $\sim \alpha$ *term of* β_κ! This result holds for *any* $SU(N)$ flavor gauge theory, the *only requirement* is that the *number of lepton generations be equal to the degree of the group*, $n \overset{!}{=} N$. Therefore, to obtain $\sim G_{(\pm)}$ terms in β_κ from flavor gauge theories, it is necessary to either have a $U(1)$ gauge group, an $SU(N)$ group where leptons are in non-fundamental representations, or perhaps an $SO(N)$ gauge group. There is one further possibility: integrating out of some of the $SU(N)$ gauge such that they do not contribute to the running of κ anymore.

Before discussing this aspect further, let us summarize the main results and most salient discussion points of this subsection thus far—note that we will include the result for fundamental representations of $SU(N)$ in the next summary.

For **interactions of the lepton-doublets with vector bosons** analogous to those arising from gauge covariant derivatives, the **resulting $\sim \boldsymbol{G}_{(\pm)}$ terms are given for any number of lepton generations, potential gauge group, and representation** by a **universal loop-integral, and coupling structure**.

The **neutral and charged contributions**, as well as the **$\boldsymbol{G}$ and $G_\pm$, are in direct correspondence to one another; always appear with a positive sign**; and are in **full generality** given by

$$\mathcal{L} \supset -g_n \, T_{gf} \, \overline{l^g} \, \gamma^\mu \, l^f \, V_\mu - \frac{1}{\sqrt{2}} \Big[g_c \, T_{+,\, gf} \, \overline{l^g} \, \gamma^\mu \, l^f \, V_\mu^+ + \text{h.c.} \Big] \,, \tag{II-2.270}$$

$$\delta\mathcal{K}_{gf,\,1} \, \Big|_{llV_\mu} = \frac{2}{16\pi^2} \, g_n^2 \, (3 + \xi_n) \quad \big(T^T \, \mathcal{K} \, T\big)_{gf} \,, \tag{II-2.271}$$

$$G = \sqrt{\frac{2}{16\pi^2} \, g_n^2 \, (3 + \xi_n)} \quad T \,; \tag{II-2.272}$$

$$\delta\mathcal{K}_{gf,\,1} \, \Big|_{llV_\mu^\pm} = \frac{2}{16\pi^2} \, g_c^2 \, (3 + \xi_c) \quad \frac{1}{2} \Big(T_+^T \, \mathcal{K} \, T_- + T_-^T \, \mathcal{K} \, T_+\Big)_{gf} \,, \tag{II-2.273}$$

$$G_\pm = \sqrt{\frac{2}{16\pi^2} \, g_c^2 \, (3 + \xi_c)} \quad T_\pm \,. \tag{II-2.274}$$

We show the interaction terms in eq. (II-2.270). The $\delta\mathcal{K}$, and the $G_{(\pm)}$ extracted from them arise from calculating the pertinent $llV_\mu^{(\pm)}$ vetrex corrections—note that we may also choose a negative prefactor for $G_{(\pm)}$.

The results are **independent of the vector bosons' masses, and gauge-dependent**.

For **any vector boson interaction, only the appropriate substitutions of coupling strengths, coupling matrices or generators, and gauge parameters are needed—this includes flavor-universal (gauge) interactions**. **For mutiple vector bosons, we sum over their individual contributions**.

II-2.3.3.1 Mass Requirements on Intermediate Vector Bosons

So far, we have calculated the contributions from intermediate vector bosons to the renormalization of κ, and thus its running. However, there is a tacit assumption necessary for this:

The masses of contributing intermediate vector bosons must be of the same order or smaller than the renormalization scale that enters the running of κ via the $\beta_{\mathcal{K}}$-function.

Note that while this is true for all contributing intermediate particles, we focus on the vector bosons, as these are the ones giving rise to the new quantum effects at the center of this work.

The requirement above stems from the following consideration: We calculate loop-diagrams that are renormalized at a certain scale μ; at this scale, we calculate the $\beta_{\mathcal{K}}$-function that determines the running of κ. The solution of the differential equation $\beta_{\mathcal{K}}$ is—essentially—$\kappa(\mu)$; however, this solution is only reasonably applicable as longs as the intermediate particles used to calculate $\beta_{\mathcal{K}}$ are still active—i.e., as long as μ is larger or of the same order as the intermediate particles' masses. If the renormalization scale now drops below the mass scale of an intermediate particle, this particle needs to be integrated out, as it becomes effectively inactive. Therefore, it does not give any additional contributions to $\beta_{\mathcal{K}}$, and does not influence the running of κ any further.

There are two messages contained in this consideration.

1. We need particles whose masses are of the same order or smaller than the renormalization scale, and
2. particles that are heavier do not contribute to $\beta_{\mathcal{K}}$ anymore.

In our context, this leads us to the realizations that

1. if we want $\sim G_{(\pm)}$ terms in $\beta_{\mathcal{K}}$, the mass scale of the intermediate vector bosons need to be of the same order or lower than the renormalization scale μ; and
2. we can obtain $\sim G_{(\pm)}$ terms from $SU(N)$ flavor gauge theories with the leptons in the fundamental representation if *some* of the gauge bosons are integrated out and do not contribute anymore.

We also note that to contribute to the renormalization of κ at all, the masses of intermediate particles need to be of the same order or lower than the UV scale Λ, up to which the effective description via κ is reasonable; above Λ the effective description breaks down and κ is not reasonably defined anymore.

Let us discuss the second point, motivated by the result of eq. (II-2.269). Namely, we notice that for the result of eq. (II-2.269) to hold, we require *all* gauge bosons be present and contribute. Therefore, based on our current discussion of the mass and renormalization scales, there is still a way we can obtain $\sim G_{(\pm)}$ terms—we only need to imagine a model in which not all gauge bosons contribute. In particular, as we have just seen, we can achieve this using the gauge boson masses, even though the results of the diagrams themselves do not depend on them. In a spontaneously broken $SU(N)$ theory, the gauge bosons may acquire masses, which do not have to be equal to one another—the most famous example being the electroweak sector of the Standard Model. This means that if we break the symmetry spontaneously, and some gauge bosons acquire masses heavier than the others, the heavier ones can be integrated out below their mass scale. Thus, the remaining gauge bosons still contribute to $\beta_{\mathcal{K}}$, while the heavier ones do not, meaning that we cannot apply the completeness relation of eq. (II-2.62) to obtain eq. (II-2.269). Therefore, we can still obtain $\sim G_{(\pm)}$ terms in $\beta_{\mathcal{K}}$ in such models! Basically, this tells us that *we cannot—or from another standpoint, we do not have to—exclude $SU(N)$ flavor gauge theories with the leptons in the fundamental representation as candidates for theories with $\sim G_{(\pm)}$ terms in $\beta_{\mathcal{K}}$*!

Let us now summarize the points about the mass scales, and our findings for the fundamental representation of $SU(N)$.

To obtain $\boldsymbol{\sim G_{(\pm)}}$ terms, **we need flavor-nonuniversally-interacting vector bosons with masses at or below the renormalization—and UV—scale**. Vectors with **masses above the renormalization scale are integrated out and do not contribute anymore**.

If **all gauge bosons of any $\boldsymbol{SU(N)}$ gauge theory with the leptons in the fundamental representation are active, they only contribute to $\boldsymbol{\alpha\,\mathcal{K}}$**, as

$$\delta\mathcal{K}_{gf,1}\Big|_{llSU(N)_{\text{fund.}}} = \frac{2}{16\pi^2}\, g^2_{SU(N)}\,(3+\xi_{SU(N)})\;\frac{N-1}{2\,N}\quad\mathcal{K}\,. \qquad \text{(II-2.275)}$$

However, **even in the fundamental representation of $\boldsymbol{SU(N)}$, $\boldsymbol{\sim G_{(\pm)}}$ terms may be obtained if some of the gauge bosons' masses are above the renormalization scale, such that the respective gauge bosons are integrated out and do not contribute to $\boldsymbol{\beta_{\mathcal{K}}}$ anymore**.

This concludes the general discussion of the $llV_\mu^{(\pm)}$ loop-topology. Before moving on to specific models, let us calculate the contribution to lepton wave function renormalization coming from the vector bosons, as this will be very useful later on.

II-2.3.3.2 Universal Contribution to Lepton Wave Function Renormalization

Previously, we have discussed the integral, and coupling structure universality of the contributions giving rise to the new quantum effect introduced in this work. In the next chapter, we will present and discuss specific models where the corresponding terms in the $\beta_{\mathcal{K}}$-function may arise. In particular, we will consider models with flavor gauge theories; therefore, it is advantageous to perform the relevant, repeatedly appearing loop-integrals for the general case and then apply them directly to the models later on. Hence, we will now calculate the second diagram that is, generally speaking, present in *any* theory with a coupling of two lepton-doublets to a vector boson—the wave function renormalization bubble of the lepton-doublet.

Since further aspects, and fields of particular models—such as scalar fields that may break a gauge symmetry spontaneously, other fermions, etc.—depend heavily on the specifics of the model itself, we will not perform all possible integrations and calculate all potentially present interactions. Instead, we are particularly interested in the $llV_\mu^{(\pm)}$ loop-topology, which necessitates a coupling of the leptons to the vector bosons as in eq. (II-2.270). Therefore, there are exactly two topologies that renormalize κ, the vertex correction $llV_\mu^{(\pm)}$, and the wave function renormalization of the lepton doublet via the vectors V_μ and $V_\mu^{\pm}$. We have already calculated and discussed the vertex correction, so we now focus on the contribution to wave function renormalization.

The diagrams giving rise to the wave function renormalization of the lepton-doublets via the vector bosons are of the form as displayed in fig. (II-2.11). These bubbles result in terms with $P \sim T^\dagger T = T^2$ and $P \sim T_\mp T_\pm = T_\pm^\dagger T_\pm$—or analogous terms in α. We use the same interactions here, as they were defined in eq. (II-2.270).

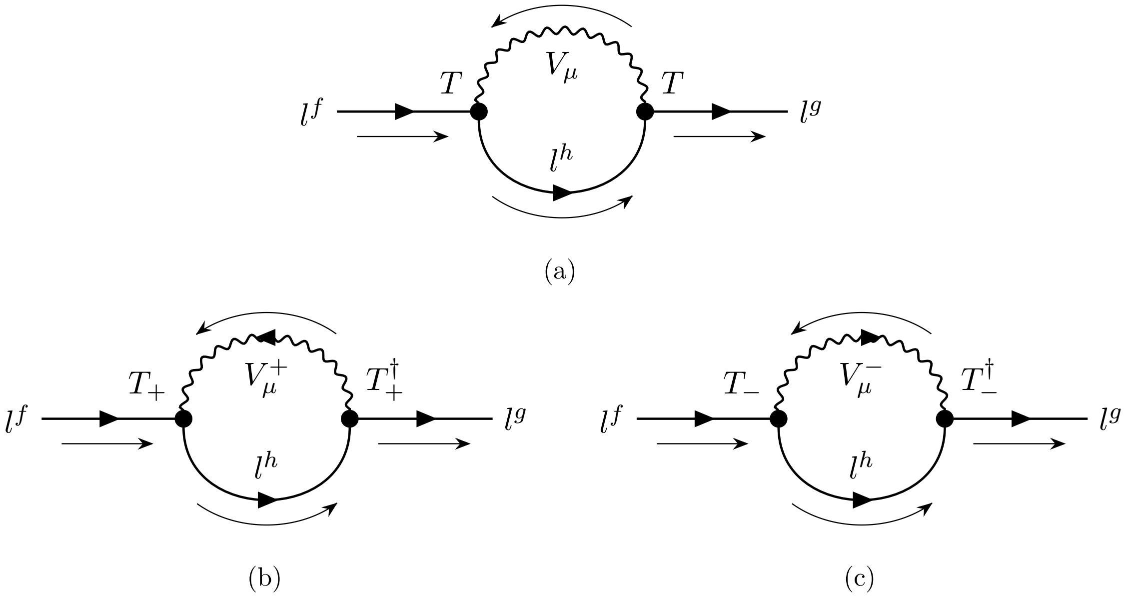
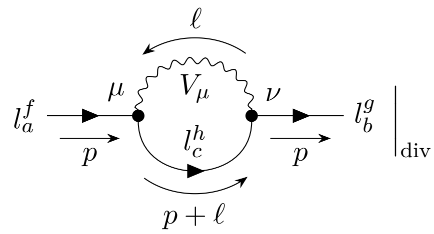


Figure II-2.11.: Contribution of some neutral vector boson V_μ, and charged vector bosons $V_\mu^\pm$ to the wave function renormalization of the lepton-doublets; the neutral vector contribution is displayed in fig. (II-2.11a), the positive, in fig. (II-2.11b), and the negative, in fig. (II-2.11c); T and $T_\pm$ are matrices in flavor-space coming from the couplings between the respective vector bosons and the lepton-doublets; $T_\pm^\dagger$ appear because the charge flow is inverted from the perspective of the right vertices; the superscript is the lepton-family index

Using the previously derived Feynman-rules of eq. (II-2.237)–(II-2.239), we obtain for the neutral vector boson



$$= \tag{II-2.276}$$

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \Big[-i\,\tilde{\mu}^{\varepsilon}\, g_n\, \boldsymbol{T_{gh}}\, \gamma_\nu\, P_L\, \delta_{bc} \Big] \frac{i\,\big(\not{p} + \not{\ell}\big)}{(p+\ell)^2} \Big[-i\,\tilde{\mu}^{\varepsilon}\, g_n\, \boldsymbol{T_{hf}}\, \gamma_\mu\, P_L\, \delta_{ca} \Big] \quad \times \tag{II-2.277}$$

$$\times \quad \frac{i}{\ell^2 - M_V} \left[-g^{\mu\nu} + (1-\xi_n)\, \frac{\ell^\mu\, \ell^\nu}{\ell^2 - \xi_n\, M_V^2} \right]$$

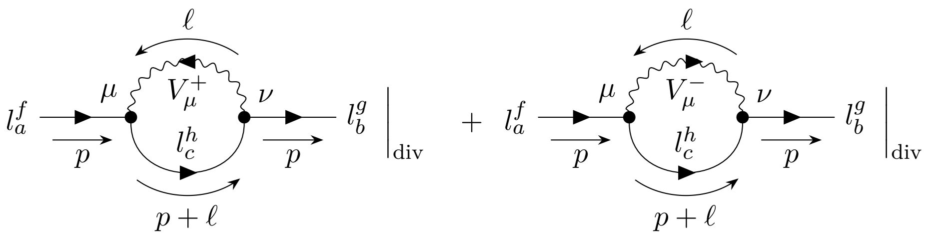
$$= -\,(-i\,g_n)^2\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\,\delta_{ba}\cdot\int\left[\mathrm{d}^d\ell\right]\gamma_\nu\,\frac{(\not{p}+\not{\ell})}{(p+\ell)^2}\,\gamma_\mu\quad\times \tag{II-2.278}$$

$$\times\quad\frac{1}{\ell^2-M_V}\left[-\,g^{\mu\nu}+(1-\xi_n)\,\frac{\ell^\mu\,\ell^\nu}{\ell^2-\xi_n\,M_V^2}\right]\cdot P_L$$

$$= -\,(-i\,g_n)^2\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\,\delta_{ba}\cdot\frac{i}{16\pi^2}\,\xi_n\,\frac{1}{\varepsilon}\,\not{p}\cdot P_L \tag{II-2.279}$$

$$= \frac{i}{16\pi^2}\,\not{p}\,P_L\,\delta_{ba}\,\frac{1}{\varepsilon}\,g_n^2\,\xi_n\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\,. \tag{II-2.280}$$

For the charged vector bosons, we obtain



$$= \tag{II-2.281}$$

$$= \int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\left[-\,\frac{i}{\sqrt{2}}\,\tilde{\mu}^\varepsilon\,g_n\,\left(\boldsymbol{T}_+^\dagger\right)_{\boldsymbol{gh}}\,\gamma_\nu\,P_L\,\delta_{bc}\right]\frac{i\,(\not{p}+\not{\ell})}{(p+\ell)^2}\left[-\,\frac{i}{\sqrt{2}}\,\tilde{\mu}^\varepsilon\,g_n\,\boldsymbol{T}_{+,\boldsymbol{hf}}\,\gamma_\mu\,P_L\,\delta_{ca}\right]\quad\times \tag{II-2.282}$$

$$\times\quad\frac{i}{\ell^2-M_V}\left[-\,g^{\mu\nu}+(1-\xi_n)\,\frac{\ell^\mu\,\ell^\nu}{\ell^2-\xi_n\,M_V^2}\right]\quad+\quad\left(+\longleftrightarrow-\right)$$

$$= -\,(-i\,g_n)^2\,\frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\boldsymbol{T}_+^\dagger\,\boldsymbol{T}_+ + \boldsymbol{T}_-^\dagger\,\boldsymbol{T}_-\right)_{\boldsymbol{gf}}\,\delta_{ba}\cdot\int\left[\mathrm{d}^d\ell\right]\gamma_\nu\,\frac{(\not{p}+\not{\ell})}{(p+\ell)^2}\,\gamma_\mu\quad\times \tag{II-2.283}$$

$$\times\quad\frac{1}{\ell^2-M_V}\left[-\,g^{\mu\nu}+(1-\xi_n)\,\frac{\ell^\mu\,\ell^\nu}{\ell^2-\xi_n\,M_V^2}\right]\cdot P_L$$

$$= -\,(-i\,g_n)^2\,\frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\boldsymbol{T}_+^\dagger\,\boldsymbol{T}_+ + \boldsymbol{T}_-^\dagger\,\boldsymbol{T}_-\right)_{\boldsymbol{gf}}\,\delta_{ba}\cdot\frac{i}{16\pi^2}\,\xi_n\,\frac{1}{\varepsilon}\,\not{p}\cdot P_L \tag{II-2.284}$$

$$= \frac{i}{16\pi^2}\,\not{p}\,P_L\,\delta_{ba}\,\frac{1}{\varepsilon}\,g_n^2\,\xi_n\,\frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\boldsymbol{T}_+^\dagger\,\boldsymbol{T}_+ + \boldsymbol{T}_-^\dagger\,\boldsymbol{T}_-\right)_{\boldsymbol{gf}}\,. \tag{II-2.285}$$

Let us now apply eq. (II-2.230), we denote this topology by $lV_\mu^{(\pm)}$. For the contribution from the neutral vector, we obtain

$$\delta Z_{l,\,gf,\,1}\,\Big|_{lV_\mu} = -\,\frac{2\varepsilon}{i\,\not{p}\,P_L\,\delta_{ba}}\cdot\mathcal{B}^{lV_\mu,\,gf}_{l,\,ba}\Big|_{\text{div}} \tag{II-2.286}$$

$$= -\,\frac{2\varepsilon}{i\,\not{p}\,P_L\,\delta_{ba}}\cdot\left[\frac{i}{16\pi^2}\,\not{p}\,P_L\,\delta_{ba}\,\frac{1}{\varepsilon}\,g_n^2\,\xi_n\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\right] \tag{II-2.287}$$

$$= -\,\frac{2}{16\pi^2}\,g_n^2\,\xi_n\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\,. \tag{II-2.288}$$

Analogously, we find for the charged vectors that

$$\delta Z_{l,\,gf,\,1}\,\Big|_{lV_\mu^\pm} = -\,\frac{2\varepsilon}{i\,\not{p}\,P_L\,\delta_{ba}}\cdot\mathcal{B}^{lV_\mu,\,gf}_{l,\,ba}\Big|_{\text{div}} \tag{II-2.289}$$

$$= -\,\frac{2\varepsilon}{i\,\not{p}\,P_L\,\delta_{ba}}\cdot\left[\frac{i}{16\pi^2}\,\not{p}\,P_L\,\delta_{ba}\,\frac{1}{\varepsilon}\,g_n^2\,\xi_n\,\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{T_+^\dagger}\,\boldsymbol{T_+}+\boldsymbol{T_-^\dagger}\,\boldsymbol{T_-}\right)_{\boldsymbol{gf}}\right] \tag{II-2.290}$$

$$= -\,\frac{2}{16\pi^2}\,g_n^2\,\xi_n\,\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{T_+^\dagger}\,\boldsymbol{T_+}+\boldsymbol{T_-^\dagger}\,\boldsymbol{T_-}\right)_{\boldsymbol{gf}}\,. \tag{II-2.291}$$

We observe that the same points made previously for the $llV_\mu^{(\pm)}$ apply in the same way here; e.g., loop-integral, and coupling structure universality, obtaining the expected structures, independence of the gauge boson mass, gauge-dependence, applicability to *any* vector boson interaction—whether flavor-universal or not—recovering the SM result, etc. Therefore, we will not discuss these aspects again, as it is completely analogous to before.

Note that these terms contribute with a *positive sign* in $\beta_{\mathcal{K}}$, due to the prefactor of $-1/2$ in front of the wave function renormalization constants.

While we will not repeat most of the previous discussions, let us consider the case of *gauge theories, where all gauge bosons are active.* In that case, the contributions to δZ_l simplify by virtue of the quadratic Casimir of the respective gauge group Γ and representation R. This applies both to flavor-universal, and flavor-nonuniversal interactions, and thus also flavor gauge theories. To see how this result comes about, we choose the neutral basis of gauge bosons, and sum over all contributions of the form in eq. (II-2.288):

$$\sum_{A=1}^{\dim(\Gamma)}\left[\left(T_R^A\right)^2\right]_{gf} = \sum_{A=1}^{\dim(\Gamma)} T^A_{R,\,gh}\,T^A_{R,\,hf} \tag{II-2.292}$$

$$\overset{\substack{\text{def. of quadr.}\\ \text{Casimir}}}{=}\quad C_R^\Gamma\,\delta_{gf}\,. \tag{II-2.293}$$

This means that if all gauge bosons are active, this topology only contributes to the $\alpha\,\kappa$ term of β_{κ}, since

$$\delta Z_{l,\,gf,\,1}\Big|_{l\Gamma} = \sum_{A=1}^{\dim(\Gamma)} \left\{ -\frac{2}{16\pi^2}\, g_n^2\, \xi_n\, \left[\left(\boldsymbol{T_R^A}\right)^{\boldsymbol{2}}\right]_{\boldsymbol{gf}} \right\} \tag{II-2.294}$$

$$= -\frac{2}{16\pi^2}\, g_n^2\, \xi_n\, C_R^\Gamma\, \boldsymbol{\delta_{gf}}\,. \tag{II-2.295}$$

Explicitly, we thus obtain for the corresponding contribution to β_{κ}

$$\Longrightarrow \quad \beta_{\kappa} \supset -\frac{1}{2}\,\delta Z_{l,1}^T\,\kappa - \frac{1}{2}\,\kappa\,\delta Z_{l,1} \overset{l\Gamma}{\supset} \underbrace{\frac{2}{16\pi^2}\, g_n^2\, \xi_n\, C_R^\Gamma}_{\equiv\, \alpha_\Gamma}\;\kappa\,, \tag{II-2.296}$$

which, as we see, yields a term $\alpha_\Gamma\,\kappa$.

Let us mention two points here. First, this consideration did not require a particular gauge group or representation, it applies to any semisimple Lie-Group, as these possess such quadratic Casimir operators—note, however, that non-compact groups would not be suitable as gauge groups, i.e., we are basically just left with $SU(N)$ and $SO(N)$. Furthermore, analogously to the $llSU(N)$ vertex correction, if some of the gauge bosons are integrated out below their mass scale, only the remaining ones still participate in the renormalization of κ, such that we cannot apply the Casimir operator. In such scenarios, we would still obtain $P^T\,\kappa + \kappa\,P$ terms in β_{κ}.

Let us now summarize the findings of this part.

In **any theory with vector bosons that interact with the lepton-doublets, wave function renormalization diagrams are present** in addition to the vertex corrections. Bearing the **same properties as the $\boldsymbol{llV_\mu^{(\pm)}}$ vertex topology**, the $\boldsymbol{lV_\mu^{(\pm)}}$ **topology** contributes to the wave function renormalization of the lepton doublets **full generality** as

$$\delta Z_{l,\,gf,\,1}\,\Big|_{lV_\mu} = -\,\frac{2}{16\pi^2}\,g_n^2\,\xi_n\,\left(\boldsymbol{T^2}\right)_{\boldsymbol{gf}}\,, \tag{II-2.297}$$

$$\delta Z_{l,\,gf,\,1}\,\Big|_{lV_\mu^\pm} = -\,\frac{2}{16\pi^2}\,g_n^2\,\xi_n\,\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{T_+^\dagger\,T_+ + T_-^\dagger\,T_-}\right)_{\boldsymbol{gf}}\,. \tag{II-2.298}$$

The **discussions concerning universality, application, etc. made for the $\boldsymbol{llV_\mu^{(\pm)}}$ topology applies here as well**.

In any gauge theory—flavor-universal, -nonuniversal, as well as flavor gauge theory—if all gauge bosons are active, this topology only contributes to the $\boldsymbol{\alpha\,\mathcal{K}}$ term in $\boldsymbol{\beta_{\mathcal{K}}}$. This fact is independent of the representation of the leptons under this gauge group. Note that **integrating out some of the (heavy) gauge bosons may invalidate this statement**. For a **gauge group $\boldsymbol{\Gamma}$** and the leptons in the **representation $\boldsymbol{R}$**, the **contribution to $\boldsymbol{\beta_{\mathcal{K}}}$** is **proportional to the corresponding quadratic Casimir opertor**, and given by

$$\beta_{\mathcal{K}}\,\Big|_{l\Gamma} \supset \frac{2}{16\pi^2}\,g_n^2\,\xi_n\,C_R^\Gamma\,\boldsymbol{\mathcal{K}} \tag{II-2.299}$$

$\boldsymbol{U(1)}$ flavor gauge theories are the only exception to this, as there is only one gauge boson and $\boldsymbol{U(1)}$ is abelian group. For **$\boldsymbol{U(1)}$ eq. (II-2.297) is applied**, and $\boldsymbol{T^2} \neq \mathbb{1}$ in general.

This concludes our discussion on the renormalization of $\mathcal{K}$ in potentially flavor-nonuniversally interacting, general (gauge) theories. In the next chapter, we will consider, and in part build, models of flavor gauge extensions of the Standard Model, and discuss them in the context of the $\beta_{\mathcal{K}}$-function.

CHAPTER

II-3

Models of Flavor Gauge Theories

So far, we have discussed the $\beta_\mathcal{K}$-function in a very general context, and derived fundamental equations governing the running of κ. Let us now apply our findings to particular models. Inspired by previous models, we will build and extend frameworks of flavor gauge theories to generate neutrino masses, and discuss the renormalization of the Weinberg-Operator in them. Over the course of this, we will discuss various generally relevant points in detail, with the models providing concrete examples.

Before starting with specific models, let us first answer the following question:

II-3.1 What Kind of Models are We Looking For?

To begin with, we would like to have some models that implement the new quantum effects discovered in this work. However, flavor gauge symmetries impose restrictions on the structure of κ—the reason being that the interaction term needs to be gauge-invariant. Such structures are thus subject to more stringent experimental constraints. On one hand, this means that we are looking for models, where the flavor symmetries are not present anymore today. Therefore, we need to break the symmetry spontaneously at some higher energy scale; in other words, we are interested in models where we can start at a high scale at which the flavor symmetries are broken, and then run the RGEs down to lower energies. However, a small breaking will only alter the structure of κ slightly—or at least not drastically. Thus, to be free to choose any initial condition for κ at the UV scale we start the running from, we aim to break the symmetry badly, essentially "washing out" the particular structure it imposes.

Furthermore, even after breaking the gauge symmetry, the gauge bosons are still active, and may thus lead to the flavor-nonuniversal, $llV_\mu^{(\pm)}$ vertex corrections that can raise the rank of the mass matrix and generate new eigenvalues. The benefit of this is that we can both escape the structural constraints on κ due to the badly broken symmetry, but still employ the nice one-loop effects of flavor gauge theories—the flavor-nonuniversally interacting vector bosons.

II-3.2 Models of Abelian Flavor Gauge Symmetries: $U(1)$

The most straight-forward flavor gauge theory we can imagine is with an abelian $U(1)$ group. This is not a new idea, as such models have been discussed in detail before, and applied, e.g., to electroweak measurements–for instance, to explain the anomalous magnetic moment of the muon [11]. In particular, as discussed in the first part, $U(1)_{L_\mu - L_\tau}$ is an interesting candidate. In the usual, non-badly broken scenario, $U(1)_{L_\mu - L_\tau}$ with an $SU(2)_L$ singlet scalar breaking the symmetry, seems to be the only remaining minimal $U(1)_{L_\alpha - L_\beta}$ model that is not yet excluded by experiments [10]. Recall that $U(1)_{L_\alpha - L_\beta}$ models are such where the difference of two flavor numbers is gauged. However, even though we will mostly focus on the $U(1)_{L_\mu - L_\tau}$ case due to its ubiquitous presence, we will also discuss general flavor $U(1)$ gauge symmetries. Given the badly broken scenario we are interested in here, this is not unreasonable, as experimental constraints on flavor structure and thus the symmetries underlying it do not apply in the same way if we assume so-called neutrino mass anarchy—seemingly random, structureless entries in the neutrino mass matrix [51].

Let us first discuss one general requirement for any flavor gauge symmetry: anomaly cancellation. Quantum Field Theories may exhibit so-called anomalies; these occur when the action of the theory is invariant under a symmetry transformation, but the path integral measure is not. Fundamentally, the presence of anomalies is in principle not a problem, and in fact part of nature—in the Standard Model the much heavier mass of the η' boson compared to the pions is the result of an anomaly of the so-called chiral symmetry of the strong sector. Fundamentally, the meaning of anomalies is that the seemingly present symmetry, is actually not a symmetry of the quantum theory. While is acceptable for global symmetries, it is disastrous for gauge symmetries, as them being a symmetry of the quantum theory is necessary for the correct quantization of the respective spin-1 particles—recall our discussions in the previous chapter. This means that it is absolutely crucial that gauge symmetries are non-anomalous.

While we will not show it here, as it is part of any advanced textbook on QFT—see, e.g., [36, 30]—the relevant object we need to consider, to check whether a gauge theory is anomalous, is the anomaly factor

$$\mathrm{Tr}\left[\gamma_5\, T^A \left\{T^B\,,\, T^C\right\}\right]. \tag{II-3.1}$$

The T^i are gauge group generators of up to three different gauge groups, or one gauge group in the case of the gravitational anomaly; the trace runs over all chiral fermion multiplets and the generators; and the γ_5 parametrizes the sign in front of the contribution—this gives -1 for left-handed fermions, and $+1$ for right-handed ones. Now, we would need to consider all combinations of the flavor gauge group with the SM gauge groups to check that the anomaly factor in eq. (II-3.1) vanishes, as a non-vanishing anomaly factor corresponds to the presence of a gauge anomaly.

However, since we are summing over left- and right-handed fermions with different signs, we already know that the factor vanishes for only vector-like symmetries—i.e., those where left-, and right-handed fermions interact in the same way. Since we assume such a vector-like symmetry for the gauged lepton flavor, the sum over fermions in the gravitational anomaly vanishes, as do those with $SU(3)_c$ of QCD. If, however, we consider the anomaly with our new gauge group, whose charge matrix we call $\tilde{Q}$, with two $SU(2)_L$ generators, we find that

$$\mathrm{Tr}\left[\gamma_5\,\tilde{Q}\,\left\{T^i_{SU(2)_L}\,,\,T^j_{SU(2)_L}\right\}\right] = \frac{1}{4}\,\mathrm{Tr}\left[\gamma_5\,\tilde{Q}\,\left\{\tau^i\,,\,\tau^j\right\}\right] \tag{II-3.2}$$

$$= \frac{1}{4}\,2\,\delta^{ij}\,\underset{\text{left-handed}}{\mathrm{Tr}}\left[\tilde{Q}\right] \tag{II-3.3}$$

$$\Longrightarrow \quad \mathrm{Tr}\left[\tilde{Q}\right] = 0\,. \tag{II-3.4}$$

This means that to avoid gauge anomalies, the trace of the charge matrix itself needs to vanish! Note that we used the fact that right-handed leptons do not couple to $SU(2)_L$. Therefore, we can parametrize any flavor $U(1)'$ gauge charge matrix as

$$\tilde{Q}_{\text{general}} = \begin{pmatrix} -\tilde{q}_\mu - \tilde{q}_\tau & 0 & 0 \\ 0 & \tilde{q}_\mu & 0 \\ 0 & 0 & \tilde{q}_\tau \end{pmatrix}\,, \tag{II-3.5}$$

where $\tilde{q}_\mu$ and $\tilde{q}_\tau$ are the charges of the second (μ), and third (τ) generation. We chose this parametrization, since it can be straightforwardly applied to $U(1)_{L_\mu - L_\tau}$ via $\tilde{q}_\mu \longrightarrow 1$, and $\tilde{q}_\tau \longrightarrow -1$.

For non-abelian flavor gauge theories, we do not get any new constraints, as they are anomaly-free by default. This is because the generators of $SU(N)$ and $SO(N)$ are trace-less by definition, and in this context they represent a vector-like symmetry, such that all the traces vanish much like for QCD or QED.

II-3.3 Understanding Renormalization of κ in $U(1)$ Flavor Gauge Theories: The Example of $U(1)_{L_\mu - L_\tau}$

Let us now go to the $U(1)_{L_\mu - L_\tau}$ gauge extension, and consider the Weinberg-Operator,

$$\sim \kappa_{gf}\, l^g\, l^f\, \phi\, \phi\,. \tag{II-3.6}$$

and its renormalization. We define the $U(1)_{L_\mu - L_\tau}$ coupling constant as $\tilde{g}$. If we now perform the $U(1)_{L_\mu - L_\tau}$ gauge transformation,

$$l \longrightarrow e^{i\,\alpha(x)\,\tilde{g}\,\tilde{Q}}\; l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{+i\,\alpha(x)\,\tilde{g}} & 0 \\ 0 & 0 & e^{-i\,\alpha(x)\,\tilde{g}} \end{pmatrix} l\,, \tag{II-3.7}$$

where l is a three-vector in flavor-space. We see that the Weinberg-Operator is actually not entirely gauge-invariant! In fact, only three entries are allowed at this level,

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{11} & 0 & 0 \\ 0 & 0 & \mathcal{K}_{23} \\ 0 & \mathcal{K}_{23} & 0 \end{pmatrix} . \tag{II-3.8}$$

Here we see the structure restrictions imposed by the symmetry, mentioned earlier. To obtain the other entries, we need to break the symmetry; the remaining diagonal entries break $U(1)_{L_\mu - L_\tau}$ by two units, the others by one—see, for instance, [11]. The new entries gained via spontaneous symmetry breaking are, however, not manifestly gauge-invariant. We will see the ramifications of this come up again later, and also discuss the issue in-depth.

Nevertheless, for now, let us ignore this fact and continue with our consideration. Let us use the formulae we derived in the previous chapter to calculate the $\beta_{\mathcal{K}}$-function in this gauge extension—note that we will ignore contributions coming from the symmetry-breaking scalars here. Note that in the following, we will drop the μ subscript when referring to specific topologies involving gauge bosons. From eq. (II-2.271) we obtain for the llZ' topology with the new gauge boson Z',

$$\delta\mathcal{K}_{gf,1}\Big|_{llZ'} = \frac{2}{16\pi^2}\,\tilde{g}^2\,(3+\tilde{\xi})\,\big(\tilde{Q}^T\,\mathcal{K}\,\tilde{Q}\big) \tag{II-3.9}$$

$$= \frac{2}{16\pi^2}\,\tilde{g}^2\,(3+\tilde{\xi}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{K}_{22} & -\mathcal{K}_{23} \\ 0 & -\mathcal{K}_{23} & \mathcal{K}_{33} \end{pmatrix} , \tag{II-3.10}$$

where we have defined the $U(1)_{L_\mu - L_\tau}$ gauge parameter $\tilde{\xi}$. For the wave function renormalization, we employ eq. (II-2.297) and find

$$\delta Z_{l,gf,1}\Big|_{lZ'} = -\frac{2}{16\pi^2}\,\tilde{g}^2\,\tilde{\xi}\,\big(\tilde{Q}^2\big)_{gf} \tag{II-3.11}$$

$$= -\frac{2}{16\pi^2}\,\tilde{g}^2\,\tilde{\xi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{II-3.12}$$

Finally, we use eq. (II-2.201), and directly obtain the corresponding $\beta_{\mathcal{K}}$-function,

$$\beta_{\kappa}\Big|_{Z'} = \delta\kappa_{gf,1}\Big|_{llZ'} - \frac{1}{2}\left[\delta Z_{l,gf,1}\Big|_{lZ'}\right]^T \kappa - \frac{1}{2}\kappa\,\delta Z_{l,gf,1}\Big|_{lZ'} \tag{II-3.13}$$

$$= \underbrace{\frac{2}{16\pi^2}\tilde{g}^2(3+\tilde{\xi})}_{\equiv V}\begin{pmatrix}0 & 0 & 0\\ 0 & \kappa_{22} & -\kappa_{23}\\ 0 & -\kappa_{23} & \kappa_{33}\end{pmatrix} + \underbrace{\frac{1}{16\pi^2}\tilde{g}^2\,\tilde{\xi}}_{\equiv W}\begin{pmatrix}0 & \kappa_{12} & \kappa_{13}\\ \kappa_{12} & 2\kappa_{22} & 2\kappa_{23}\\ \kappa_{13} & 2\kappa_{23} & 2\kappa_{33}\end{pmatrix} \tag{II-3.14}$$

$$= \begin{pmatrix}0 & W\,\kappa_{12} & W\,\kappa_{13}\\ W\,\kappa_{12} & \big(V+2W\big)\,\kappa_{22} & \big(-V+2W\big)\,\kappa_{23}\\ W\,\kappa_{13} & \big(-V+2W\big)\,\kappa_{23} & \big(V+2W\big)\,\kappa_{33}\end{pmatrix}. \tag{II-3.15}$$

Note that we have dropped the SM and $U(1)_{L_\mu - L_\tau}$-breaking scalar contributions, and defined the prefactors V and W for ease of notation. There are a few points we will discuss concerning this result. First, let us consider where the different signs come from, diagrammatically. For this, let us consider the $\mu\mu$ and $\mu\tau$ vertex corrections, as well as the contributions from the $e\mu$ and $\mu\mu$ wave function renormalization. We show the corresponding diagrams in fig. (II-3.1).

As we see from fig. (II-3.1), the signs originate from the charges of the fields under $U(1)_{L_\mu - L_\tau}$, which enter their coupling to the Z'. Thus, the $\mu\tau$ vertex corrections obtain a negative sign, $(+\tilde{g})\cdot(\tilde{g}) = -\tilde{g}^2$, the ones with $\mu\mu$ or $\tau\tau$ a positive sign, $(\pm\tilde{g})^2 = +\tilde{g}^2$. For the same reason, the wave function renormalization contributions $\mu\mu$ and $\tau\tau$ are always positive. Furthermore, the entries with one electron doublet cannot participate in vertex renormalization, as the Z' does not couple to them; for the wave function renormalization, the only contribution comes from the μ or τ legs. Thus, the $e\mu$ and $e\tau$ entries get factors of 1, the ee entry 0, as the Z' cannot couple to either leg. The entries with no e obtain factors of 2, since the Z' can renormalize both legs.

Now that we understand the origin of the full structure of β_{κ} in eq. (II-3.15), let us inspect the much-anticipated rank-raising ability. To that end, we consider the determinant of β_{κ}, and as we will see, the $G^T\,\kappa\,G$ term is precisely the one necessary to be able to raise the rank of β_{κ}. Note that since $\beta_{\kappa} = \mathrm{d}\kappa/\mathrm{d}t$, raising the rank of β_{κ} corresponds to raising the rank of κ, as $\kappa(t+\delta t) \approx \kappa(t) + \mathrm{d}\kappa/\mathrm{d}t\,\delta t = \kappa(t) + \beta_{\kappa}\,\delta t$. We can also make use of the identity

$$\ln\det M = \mathrm{Tr}\,\ln M\,, \tag{II-3.16}$$

and assuming β_{κ} to be invertible, i.e., of full rank. From this, we obtain

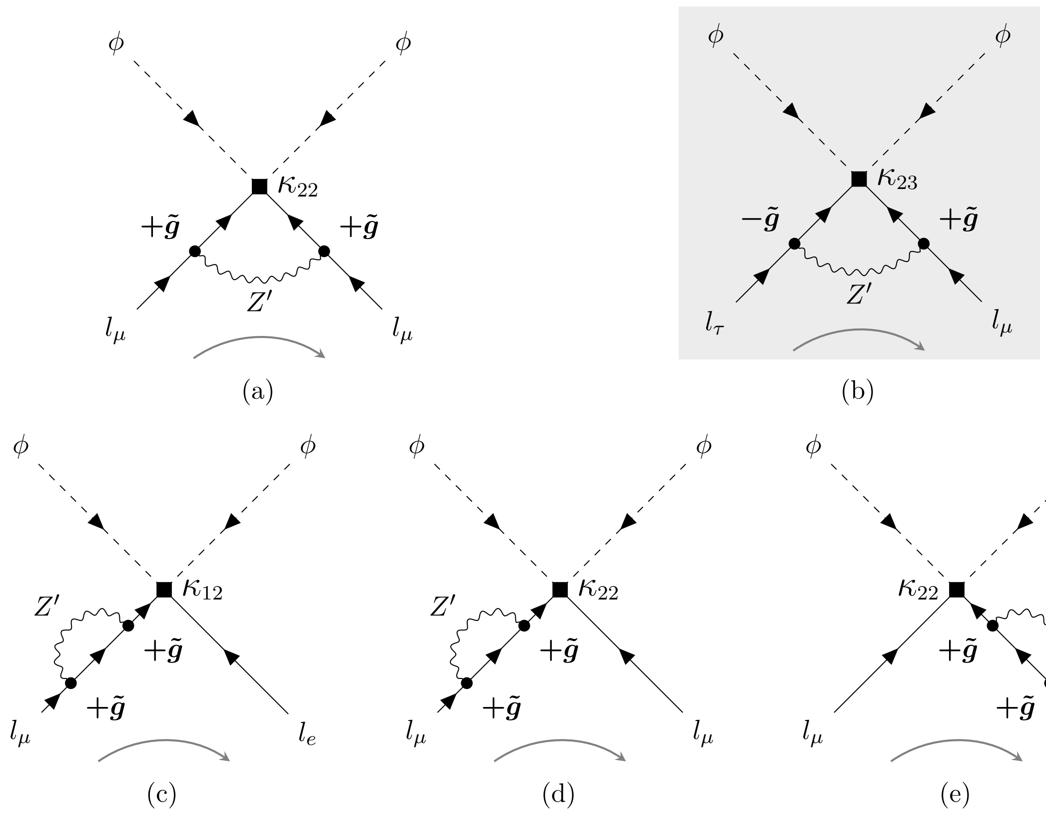


Figure II-3.1.: Contributions of Z' to the renormalization of κ via vertex corrections and wave function renormalization; the coupling constants with their signs—corresponding to the fields' charges—are shown to emphasize the origin of the signs, and the factors of two in $\beta_{\mathcal{K}}$; fig. (II-3.1a) shows the vertex correction to the $\mu\mu$ entry of $\beta_{\mathcal{K}}$, fig. (II-3.1b) to $\mu\tau$, fig. (II-3.1c) the wave function renormalization contribution to the $e\mu$ entry of $\beta_{\mathcal{K}}$, fig. (II-3.1d) and (II-3.1e) to $\mu\mu$; the gray arrows denote fermion flow

$$\det \mathcal{K}(t+\delta t) \approx \exp\left\{ \operatorname{Tr}\ln\left(\mathcal{K}(t) + \beta_{\mathcal{K}}\,\delta t\right)\right\} \tag{II-3.17}$$

$$= \exp\left\{ \operatorname{Tr}\ln\left(\frac{\mathcal{K}(t)}{\delta t}\;\beta_{\mathcal{K}}^{-1} + 1\right) + \operatorname{Tr}\ln\left(\delta t \beta_{\mathcal{K}}\right)\right\} \tag{II-3.18}$$

$$= \exp\Bigg\{ \underbrace{\underbrace{\ln\det\left(\frac{\mathcal{K}(t)}{\delta t}\;\beta_{\mathcal{K}}^{-1} + 1\right)}_{\neq 0} + \underbrace{\ln\det\left(\delta t\,\beta_{\mathcal{K}}\right)}_{\neq 0}}_{\text{finite}} \Bigg\} \neq 0\,, \tag{II-3.19}$$

where we used that for any finite value of δt, the first term does not vanish, even if $\det \mathcal{K}(t) = 0$. Note that this is only meant to illustrate the potential rank-raising ability of $\beta_{\mathcal{K}}$, and not a rigorous proof. However, let us now take a different viewpoint. We go back to eq. (II-3.15), but add an $\alpha\,\mathcal{K}$ term, and keep the prefactors V and W general. Now, we consider $\mathcal{K}(t+\delta t)$, and absorb the δt in the prefactors:

$$\mathcal{K}(t+\delta t) = \mathcal{K} + \alpha\,\mathcal{K} + W\left(\tilde{Q}^2\,\mathcal{K} + \mathcal{K}\,\tilde{Q}^2\right) + V\,\tilde{Q}^T\,\mathcal{K}\,\tilde{Q} \tag{II-3.20}$$

$$= \begin{pmatrix} (1+\alpha)\,\mathcal{K}_{11} & (1+\alpha+W)\,\mathcal{K}_{12} & (1+\alpha+W)\,\mathcal{K}_{13} \\ (1+\alpha+W)\,\mathcal{K}_{12} & (1+\alpha+V+2W)\,\mathcal{K}_{22} & (1+\alpha-V+2W)\,\mathcal{K}_{23} \\ (1+\alpha+W)\,\mathcal{K}_{13} & (1+\alpha-V+2W)\,\mathcal{K}_{23} & (1+\alpha+V+2W)\,\mathcal{K}_{33} \end{pmatrix}. \tag{II-3.21}$$

We can now calculate the determinant of this matrix, and expand it to linear order in α, W, and V; we thus obtain

$$\det \mathcal{K}(t+\delta t) \approx \det \mathcal{K} + 3\alpha \cdot \det \mathcal{K} + 4W \cdot \det \mathcal{K} + V \cdot f(\mathcal{K}_{ij}) \tag{II-3.22}$$

$$\overset{\det \mathcal{K}=0}{=} V \cdot \underbrace{f(\mathcal{K}_{ij})}_{\neq 0} \propto V\,, \tag{II-3.23}$$

where $f(\mathcal{K}_{ij})$ is a function of the entries of $\mathcal{K}$ that is *different* from $\det \mathcal{K}$. We see that the determinant of $\mathcal{K}$ after running is directly proportional to V! This means that for a nonzero $f(\mathcal{K}_{ij})$, the presence of V means that the rank of $\mathcal{K}$ is raised! And since we know that V stems from the llZ' loop-topology, we know that this flavor-nonuniversal gauge interaction is precisely what causes the increase in rank of $\mathcal{K}$, and thus the creation of new, nonzero mass eigenvalues. And since we expanded up to linear order in α, W, and V, we know that this happens at the order $\tilde{g}^2$, which is the lowest possible order for contributions from quantum effects!

Let us also consider what the determinant of $\mathcal{K}(t+\delta t)$ looks like if we are in the unbroken phase, i.e., only $\mathcal{K}_{11}$ and $\mathcal{K}_{23}$ are different from zero. In this case, we obtain

$$\det \mathcal{K}(t+\delta t) = \underbrace{-\mathcal{K}_{11}\,\mathcal{K}_{23}^2}_{=\det \mathcal{K}}\,(1+3\alpha+4W+2V)\,, \tag{II-3.24}$$

where we see that all contributions from quantum corrections are equally proportional to the determinant of $\mathcal{K}$. Note that in the symmetric phase, we have one individual eigenvalue, $\mathcal{K}_{11}$, and two degenerate ones, $\pm\mathcal{K}_{23}$. Since the entire expression in eq. (II-3.24) is proportional to the determinant of $\mathcal{K}$, we can neither increase the rank of the mass matrix in case $\mathcal{K}_{11} = 0$ or $\mathcal{K}_{23} = 0$, nor break the degeneracy of the eigenvalues $\pm\mathcal{K}_{23}$. This means that we need the entries induced by symmetry breaking to raise the rank of the mass matrix, or break the degeneracy of mass eigenvalues. Nevertheless, we have seen how the llZ' topology can, in general, raise the rank of the mass matrix—note that the crucial point is the differing signs in the entries $+V\,\mathcal{K}_{22,\,33}$ and $-V\,\mathcal{K}_{23}$.

II-3.3.1 Solving the $U(1)_{L_\mu - L_\tau}$ $\beta_{\mathcal{K}}$-Function

Let us now discuss how to solve the $\beta_{\mathcal{K}}$-function in this class of models, represented by $U(1)_{L_\mu - L_\tau}$. As we have seen in eq. (II-3.15), even in $U(1)$ flavor gauge theories, the $\beta_{\mathcal{K}}$-function of each entry is still proportional only to itself—we can understand this from the fact that $U(1)$ generators are diagonal, and thus cannot mix entries. However, without the presence of the flavor-nonuniversal llZ' topologies, we can solve the $\beta_{\mathcal{K}}$-function as a matrix differential equation. In the absence of $\sim G_{(\pm)}$ terms, the $\beta_{\mathcal{K}}$function is given by

$$\beta_{\mathcal{K}} = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = \alpha(t)\,\mathcal{K}(t) + P^T(t)\,\mathcal{K}(t) + \mathcal{K}(t)\,P(t)\,, \tag{II-3.25}$$

as we have shown in the previous chapter. We can then integrate this, and write the result in a compact form using matrix exponentials—we generalize the integration of [52, 39] to arbitrary α and P. Following [39], we define

$$I_\alpha(t) = \exp\left(-\int_t^{t_\Lambda} \mathrm{d}t'\,\alpha(t')\right), \quad I_P(t) = \exp\left(-\int_t^{t_\Lambda} \mathrm{d}t'\,P(t')\right), \tag{II-3.26}$$

with which we can write the solution of the differential eq. (II-3.25) as

$$\mathcal{K}(t) = I_\alpha(t)\,I_P^T(t)\,\mathcal{K}(t_\Lambda)\,I_P(t)\,. \tag{II-3.27}$$

By taking the derivative and applying the product rule, we can straightforwardly verify that we obtain eq. (II-3.25) again. *However*, if we add a term

$$\beta_{\mathcal{K}} \supset G^T(t)\,\mathcal{K}(t)\,G(t) \tag{II-3.28}$$

to the $\beta_{\mathcal{K}}$-function, we cannot solve the differential equation in this way anymore! Considering the structure of the matrix product, we know that even if we define an

$$I_G(t) = \exp\left(- \int_t^{t_\Lambda} \mathrm{d}t'\, G(t') \right), \tag{II-3.29}$$

we cannot get two factors of $G(t)$ via just one differentiation of an integral. Since we need one factor on either side of $\kappa(t)$, we would require at least two differentiations:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, I_G(t)^T\, \kappa(t_\Lambda)\, I_G(t) \supset G^T(t)\, \kappa(t)\, G(t) \tag{II-3.30}$$

This, however, does not fit the definition of β_κ. Therefore, we cannot employ a matrix exponential solution of β_κ, and need to resort to solving it component-wise.

We can solve eq. (II-3.15), for each component via separation of variables, and can also include an $\alpha\,\kappa$ term. Note that we can also include a general, diagonal P matrix in this consideration by using different W in the entries for β_κ:

$$P \overset{!}{=} \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{II-3.31}$$

$$\Longrightarrow P^T\,\kappa + \kappa\, P = \begin{pmatrix} 2p_1\,\kappa_{11} & (p_1+p_2)\,\kappa_{12} & (p_1+p_3)\,\kappa_{13} \\ (p_1+p_2)\,\kappa_{12} & 2p_2\,\kappa_{22} & (p_2+p_3)\,\kappa_{23} \\ (p_1+p_3)\,\kappa_{13} & (p_2+p_3)\,\kappa_{23} & 2p_3\,\kappa_{33} \end{pmatrix} \tag{II-3.32}$$

This is, for instance, the case in the Standard Model with the assumption of diagonal charged lepton Yukawa matrices—recall that in the SM $P \sim \left(Y_e^\dagger\, Y_e\right)$. We also note that with a general, hermitian P, the independent, component-wise solution of β_κ may not work anymore due to the off-diagonal mixing. To include the diagonal P case, we split the total P into the contribution from the diagonal matrix, and the W coming from the Z' contribution. We thus obtain

$$\kappa_{ij}(t) = \begin{cases} \kappa_{11}(t) = \exp\left(- \int_t^{t_\Lambda} \mathrm{d}t' \left[\alpha(t') + 2p_1(t')\right] \right) \kappa_{11}(t_\Lambda) & \\ \kappa_{1i}(t) = \exp\left(- \int_t^{t_\Lambda} \mathrm{d}t' \left[\alpha(t') + p_1(t') + p_i(t') + W(t')\right] \right) \kappa_{1i}(t_\Lambda)\,, & i = 2,\, 3 \\ \kappa_{ii}(t) = \exp\left(- \int_t^{t_\Lambda} \mathrm{d}t' \left[\alpha(t') + V(t') + 2p_i(t') + 2W(t')\right] \right) \kappa_{ii}(t_\Lambda)\,, & i = 2,\, 3 \\ \kappa_{23}(t) = \exp\left(- \int_t^{t_\Lambda} \mathrm{d}t' \left[\alpha(t') - V(t') + p_2(t') + p_3(t') + 2W(t')\right] \right) \kappa_{23}(t_\Lambda)\,. & \end{cases} \tag{II-3.33}$$

Before moving on to a concrete model for the $U(1)_{L_\mu - L_\tau}$ gauge extension, including symmetry-breaking scalars, let us investigate the β_κ-function in the two-neutrino case.

II-3.3.2 Solving the $U(1)_{L_\mu - L_\tau}$ β_κ-Function in the Two Flavor Case

If we only consider two neutrino generations in a $U(1)_{L_\mu - L_\tau}$ gauge extension, we only have the reduced charge matrix

$$\tilde{Q}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{II-3.34}$$

Furthermore, in this reduced model, we can solve the RGEs for the eigenvalues just as in the three flavor model we considered in the chapter (II-1). However, we can now also solve the RGEs for the mixing angle explicitly, and analytically, in a straightforward manner. Let us do this to see how the salient feature of this framework emerge. We will use eq. (II-2.140)–(II-2.142) combined with our previous findings in the $U(1)_{L_\mu - L_\tau}$ case. In particular, we will assume a real mixing matrix for simplicity—in this case, it is given by an $SO(2)$ matrix parametrized by the angle θ.

To begin, we define the let us gather the important quantities,

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} , \quad G = \sqrt{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad P = W \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{II-3.35}$$

From these, we obtain the transformed quantities

$$\tilde{G} = U\, G\, U^T = \sqrt{V} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} , \quad \tilde{P} = U\, P\, U^T = W \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = P , \tag{II-3.36}$$

where P remains unchanged, since $P \propto \mathbb{1}_2$. Plugging this into eq. (II-2.140), we find the coupled, linear differential equations

$$\frac{\mathrm{d}\kappa_1}{\mathrm{d}t} = \big(2W + V\, \cos^2 2\theta\big)\kappa_1 + V\, \sin^2 2\theta\; \kappa_2 , \tag{II-3.37}$$

$$\frac{\mathrm{d}\kappa_2}{\mathrm{d}t} = \big(2W + V\, \cos^2 2\theta\big)\kappa_2 + V\, \sin^2 2\theta\; \kappa_1 . \tag{II-3.38}$$

As before, we see that we raise the rank of the mass matrix, since the RGEs are coupled. Thus, if one of the eigenvalues if different from zero, the second one will be induced by virtue of the RGEs. As this is similar to what we presented in chapter (II-1), we will not go into much more detail about this here. However, we now also see another crucial aspect emerge; namely that the mixing angle needs to be different from zero. If the mixing angle is zero, the prefactors of the other eigenvalue in the RGEs, respectively, is zero, and thus we cannot induce a new, nonzero eigenvalue. As we will see shortly, this necessitates that the mixing angle at the UV scale is different from zero because otherwise, it will stay zero throughout its RGE evolution.

Let us inspect how the lifting of the mass matrix's rank happens from the point of view of the determinant in this reduced model. As in the three flavor case, we can write κ after RGE evolution as

$$\mathcal{K}(t) = \begin{pmatrix} (1+V+2W)\,\mathcal{K}_{11} & (1-V+2W)\,\mathcal{K}_{12} \\ (1-V+2W)\,\mathcal{K}_{12} & (1+V+2W)\,\mathcal{K}_{22} \end{pmatrix}, \tag{II-3.39}$$

where the determinant is given by

$$\det \mathcal{K}(t) = (1+4W)\underbrace{\left(\mathcal{K}_{11}\,\mathcal{K}_{22} - \mathcal{K}_{12}^2\right)}_{=\det \mathcal{K}(t_\Lambda)} + 2V\underbrace{\left(\mathcal{K}_{11}\,\mathcal{K}_{22} + \mathcal{K}_{12}^2\right)}_{=\det \mathcal{K}(t_\Lambda)+2\mathcal{K}_{12}}. \tag{II-3.40}$$

If now at Λ, $\det \mathcal{K}(t_\Lambda) = 0$ and $\mathcal{K}(t_\Lambda)$ is of rank one, the first term vanishes. This means that to get a $\mathcal{K}(t)$ of rank 2, i.e., with two nonzero eigenvalues, both V, and $\mathcal{K}_{12}$ need to be different from zero! Let us now write $\mathcal{K}(t_\Lambda)$ in terms of its eigenvalues—one of which is zero—and mixing matrix,

$$\mathcal{K}(t_\Lambda) = U\,\mathcal{K}_{\text{diag}}\,U^T = \begin{pmatrix} \cos\theta_\Lambda & -\sin\theta_\Lambda \\ \sin\theta_\Lambda & \cos\theta_\Lambda \end{pmatrix} \begin{pmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{pmatrix} \begin{pmatrix} \cos\theta_\Lambda & \sin\theta_\Lambda \\ -\sin\theta_\Lambda & \cos\theta_\Lambda \end{pmatrix}. \tag{II-3.41}$$

From this, we get that

$$\mathcal{K}_{12} = (\mathcal{K}_1 - \mathcal{K}_2)\,\cos\theta_\Lambda\,\sin\theta_\Lambda\,. \tag{II-3.42}$$

We thus see that also from the viewpoint of the determinant, the mixing angle at the UV scale Λ needs to be different from zero, $\theta_\Lambda \neq 0$, to be able to raise the rank of the mass matrix!

Let us now see how this happens from the point of view of the RGE for θ itself. To derive the RGE of the mixing angle, we employ eq. (II-2.142), and obtain

$$T_{11} = T_{22} = 0\,, \quad \text{(all quantities are real)}, \tag{II-3.43}$$

and

$$T_{12} = -T_{21} = \frac{\mathcal{K}_1 + \mathcal{K}_2}{\mathcal{K}_1 - \mathcal{K}_2}\cdot 0 + V\,\cos 2\theta\,\sin 2\theta\,\frac{\mathcal{K}_1}{\mathcal{K}_1 - \mathcal{K}_2} + \sin 2\theta\,(-\cos 2\theta)\,\frac{\mathcal{K}_2}{\mathcal{K}_1 - \mathcal{K}_2} \tag{II-3.44}$$

$$= \frac{1}{2}\,V\,\sin 4\theta\,. \tag{II-3.45}$$

Using eq. (II-2.141), we can then determine $\mathrm{d}\theta(t)/\mathrm{d}t$ from

$$\frac{\mathrm{d}U}{\mathrm{d}t} = T\,U = \frac{1}{2}\,V\,\sin 4\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{1}{2}\,V\,\sin 4\theta \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \tag{II-3.46}$$

$$= \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix} \frac{\mathrm{d}\theta}{\mathrm{d}t}\,, \tag{II-3.47}$$

which thus yields

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{2}\,V\,\sin 4\theta\,. \tag{II-3.48}$$

We can solve this differential equation using separation of variables and finally obtain

$$\theta(t) = \frac{1}{2} \arctan\left[\tan 2\theta_\Lambda \exp\Big(2\,V\,\big(t_\Lambda - t\big)\Big)\right]. \quad \text{(II-3.49)}$$

Let us now discuss this result. First, we observe that as anticipated before, for the values of θ at the UV scale Λ, $\theta_\Lambda = 0$, $\theta(t)$ *remains zero*. We see this by taking $\theta_\Lambda \longrightarrow 0$, which means that the $\tan 2\theta_\Lambda$ vanishes, and thus also the arctan, independently of t. Therefore, if the neutrinos have zero mixing at the scale Λ, they will not induce mixing via the RGEs, and neither an additional eigenvalue. *A nonzero mixing angle is an absolute requirement to induce new, nonzero mass eigenvalues via the new RGE effect.* Second, we see that when we run t down, the argument of the exponential function, $\sim t_\Lambda - t$ *increases*, this means that the exponential function increases, and thus the arctan. Therefore, since $V > 0$, we know that the mixing angle will always increase! In fig. (II-3.2), we show the running of $\theta(t)$ for various values of θ_Λ for $\Lambda \sim 10^{14}$ GeV. As we can see, all mixing angles increase, although by different amounts. We understand this from the fact that changing θ_Λ changes the prefactor of the exponential in the argument of the arctan, and thus the amount the running of t changes $\theta(t)$.

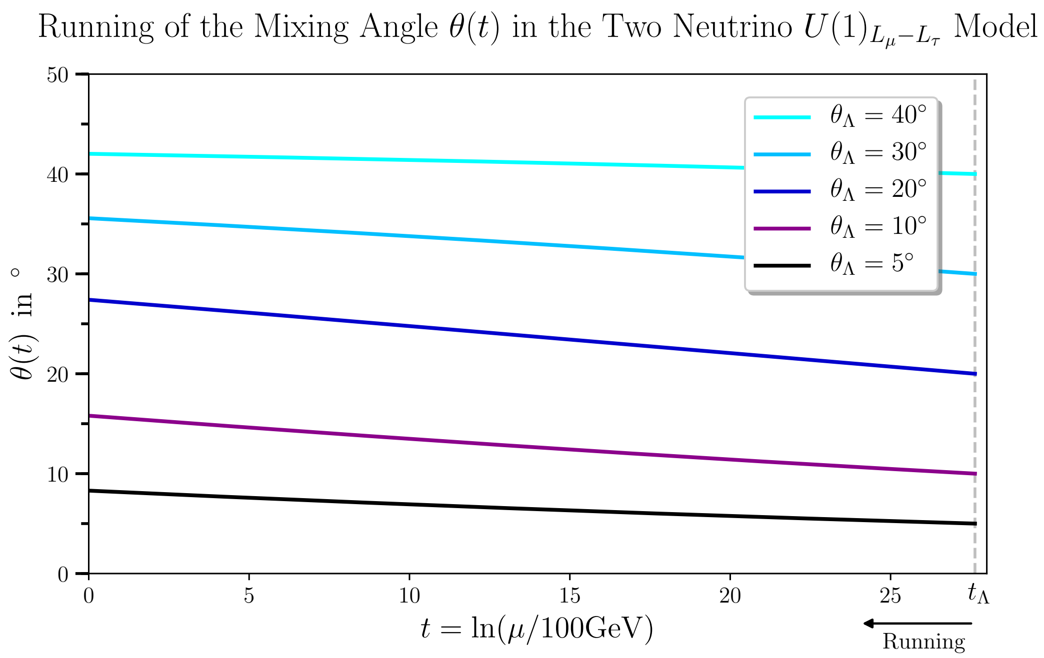


Figure II-3.2.: Running of the mixing angle $\theta(t)$ due to the vertex renormalization effects of Z', as a function of t; we run the RGEs down from $t_\Lambda = \ln(\Lambda/100\,\text{GeV})$, with $\Lambda = 10^{14}$ GeV, $\tilde{g} = 0.5$, and $\tilde{\xi} = 0$; at t_Λ, we set five different boundary values for $\theta(t_\Lambda) = \theta_\Lambda$, 5°, 10°, 20°, 30°, and 40°

Let us now summarize, and partially generalize the findings of this section thus far.

Due top the requirement of gauge anomaly cancelation, the **charge matrix of any $\boldsymbol{U(1)}$ flavor gauge theory needs to be traceless**. In $\boldsymbol{SU(N)}$ and $\boldsymbol{SO(N)}$ **flavor gauge theories, no constraint is imposed** to realize gauge anomaly cancelation. The **most general charge matrix for a $\boldsymbol{U(1)}$ favor gauge theory** can be parametrized as follows, with the resulting **entries of $\boldsymbol{\beta_{\mathcal{K}} \sim (\tilde{Q}^T \kappa \tilde{Q}^T)}$**, and $\boldsymbol{\beta_{\mathcal{K}} \sim (\tilde{Q}^2 \kappa + \kappa \tilde{Q}^2)}$ being given by

$$\tilde{Q} = \begin{pmatrix} -\tilde{q}_\mu - \tilde{q}_\tau & 0 & 0 \\ 0 & \tilde{q}_\mu & 0 \\ 0 & 0 & \tilde{q}_\tau \end{pmatrix} , \tag{II-3.50}$$

$$\begin{aligned} \beta_{\mathcal{K},gf} \Big|_{llZ'} &= \frac{2}{16\pi^2} \tilde{g}^2 (3 + \tilde{\xi}) \left(\tilde{Q}^T \mathcal{K} \tilde{Q}\right)_{gf} \\ &= \frac{2}{16\pi^2} \tilde{g}^2 (3 + \tilde{\xi}) \tilde{Q}_{gg} \tilde{Q}_{ff} \mathcal{K}_{gf} , \end{aligned} \tag{II-3.51}$$

$$\begin{aligned} \beta_{\mathcal{K},gf} \Big|_{lZ'} &= \frac{1}{16\pi^2} \tilde{g}^2 \tilde{\xi} \left(\tilde{Q}^2 \mathcal{K} + \mathcal{K} \tilde{Q}^2\right)_{gf} \\ &= \frac{1}{16\pi^2} \tilde{g}^2 \tilde{\xi} \left(\tilde{Q}^2_{gg} + \tilde{Q}^2_{ff}\right) \mathcal{K}_{gf} \end{aligned} \tag{II-3.52}$$

In the following, we define the prefactors of the couplings structure as V for the llZ', and W for the lZ' topologies for brevity. Note that the **formulae presented here can be straightforwardly extended to arbitrary numbers of lepton generations**.

As discussed in the previous chapter, the presence of the llZ' loop-topology is critical to the rank-raising ability of the flavor gauge theory.

In the **presence of such contributions, $\boldsymbol{\beta_{\mathcal{K}}}$ cannot be solved with matrix exponentials**, and for **diagonal $\boldsymbol{P}$, it can be solved component-wise**. Defining p_1, p_2, and p_3 as the diagonal entries of P, we find

$$\begin{aligned} \mathcal{K}_{gf}(t) = \exp\Bigg(- \int_t^{t_\Lambda} \mathrm{d}t' \Big[& \alpha(t') + \tilde{Q}_{gg} \tilde{Q}_{ff} V(t') + \\ & + p_g(t') + p_f(t') + \left(\tilde{Q}^2_{gg} + \tilde{Q}^2_{ff}\right) W(t') \Big] \Bigg) \mathcal{K}_{gf}(t_\Lambda) . \end{aligned} \tag{II-3.53}$$

Generally speaking, **to generate new, nonzero mass eigenvalues, we require flavor mixing between massive and massless states**. Thus, to generate **three masses from just one, we need at least two nonzero mixing anlges**. **Overall, to generate a total of n massive states from m initial masses**, we **need at least $n - m$ nonzero mixing angles**.

In the **two flavor case** of $\boldsymbol{U(1)}$, the **charge matrix is given by diagonal entries of $\pm$ due to anomaly cancelation, and since an overall factor can be absorbed in the gauge coupling strength**. In this model, the **RGE for the mixing angle can be solved analytically. A nonzero mixing angle at the UV scale**, $\boldsymbol{\theta(t_\Lambda) = \theta_\Lambda \neq 0}$ is **crucial to generate new, nonzero mass eigenvalues**.

By virtue of the RGE running, the mixing angle always increases, provided it is nonzero at $\boldsymbol{\Lambda}$—note that **point applies** to the case where $\boldsymbol{p_1 = p_2 = 0}$, the situation becomes more complicated if this is not the case. The soltuion $\boldsymbol{\theta(t)}$ under this assumtion, is given by

$$\theta(t) = \frac{1}{2} \arctan \left[\tan 2\theta_\Lambda \, \exp\Big(2\, V \left(t_\Lambda - t \right) \Big) \right] . \tag{II-3.54}$$

The **RGEs for the eigenvalues and the mixing angle in the general case $\boldsymbol{p_1, p_2 \neq 0}$** are given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{2} \left[V \sin 4\theta + \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \big(p_1 - p_2 \big) \sin 2\theta \right] , \tag{II-3.55}$$

$$\frac{\mathrm{d}\kappa_1}{\mathrm{d}t} = \Big[\alpha + 2p_1 \cos^2\theta + 2p_2 \sin^2\theta + 2W + V \cos^2 2\theta \Big] \kappa_1 + + V \sin^2 2\theta \; \kappa_2 \,, \tag{II-3.56}$$

$$\frac{\mathrm{d}\kappa_2}{\mathrm{d}t} = \Big[\alpha + 2p_2 \cos^2\theta + 2p_1 \sin^2\theta + 2W + V \cos^2 2\theta \Big] \kappa_2 + + V \sin^2 2\theta \; \kappa_1 \,. \tag{II-3.57}$$

II-3.4 UV Completion of Gauged $U(1)_{L_\mu - L_\tau}$: The Six-Scalar Model

After having discussed some general aspects of renormalizing κ in $U(1)$ flavor gauge extensions—in particular $U(1)_{L_\mu - L_\tau}$—let us now consider concrete models to realize the $U(1)_{L_\mu - L_\tau}$ gauge symmetry. In particular, when investigating UV completions, we will work in a Type-I Seesaw framework. This means we will introduce right-handed, SM-singlet neutrinos that couple to the lepton-doublets via a Yukawa coupling with the Higgs-doublet.

First, let us consider the scalars we may add to the model. In this work, we focus on SM singles scalars, meaning that they do not couple to the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge group. We observe that in this case, due to the gauge representations of the left-handed lepton-doublets, we cannot build any Lorentz-invariant interaction terms with such scalars. The reasons for this are similar as to why the llV topologies can only be achieved via vector bosons. A coupling $\sim \bar{l}\, l$ that would be an SM singlet, is not Lorentz-invariant if we do not have a γ^μ in-between the doublets, which would make it a Lorentz vector. However, the Lorentz-invariant coupling $\overline{l^C}\, l$ is not gauge-invariant under the SM. Therefore, the singlet scalar cannot couple to the lepton-doublet; it may, however, couple to the Higgs-doublet. Furthermore, we assume a Type-I Seesaw mechanism with right-handed neutrinos, and assume that they couple to the SM singlet scalars.

Second, recall that we are interested in models with badly broken flavor symmetry, meaning that we want to be able to put arbitrary values in the entries of κ at the UV scale. As we have seen before, the symmetry-allowed form of κ is given by only the κ_{11} and κ_{23} entries, with the other entries necessarily being induced by symmetry breaking. Combining these two points, we propose the following model of six SM-singlet scalars S_{ij}, where the subscripts indicate to which flavors they couple—this also fixes their flavor-charge. We furthermore introduce the right-handed neutrinos N_i, whose symmetry-allowed mass matrix has the same structure as that of κ. We assume the Yukawa coupling between N_i and l_i to be diagonal in flavor-space. The Lagrangian is given by

$$-\mathcal{L} \supset Y_{\nu,i}\, \overline{l_i}\, \epsilon\, \phi^*\, N_i + \frac{1}{2}\, \lambda_{ij}\, S_{ij}\, \overline{N_i^C}\, N_j + \frac{1}{2}\, M_e e\, \overline{N_1^C}\, N_1 + M_{\mu\tau}\, \overline{N_2^C}\, N_3 \; + \text{h.c.}\,. \qquad \text{(II-3.58)}$$

In terms of mass matrices, after breaking the symmetries, this would give us

$$-\mathcal{L}_{\text{mass}} \supset \begin{pmatrix} \overline{\nu_e} & \overline{\nu_\mu} & \overline{\nu_\tau} \end{pmatrix} \underbrace{\begin{pmatrix} \frac{Y_{v,\,e}\, v_\phi}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{Y_{v,\,\mu}\, v_\phi}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{Y_{v,\,\tau}\, v_\phi}{\sqrt{2}} \end{pmatrix}}_{M_{\text{Dirac}} \equiv M_D} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} + \\ + \frac{1}{2} \begin{pmatrix} \overline{N_1^C} & \overline{N_2^C} & \overline{N_3^C} \end{pmatrix} M_R \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} . \tag{II-3.59}$$

We see that as usual, a Dirac mass matrix coupling right- and left-handed neutrinos emerges after electroweak symmetry breaking, and a right-handed Majorana mass matrix. If we integrate out the right-handed neutrinos, we obtain the Weinberg-Operator withe κ in terms of the above mass matrices. However, before writing down the form of M_R, let us consider again the scalars we added; in particular their $U(1)_{L_\mu - L_\tau}$ charges:

$$\begin{aligned} S_{ee} &\sim 0 && \longleftrightarrow && S_{\mu\tau} && \sim\ 0 \\ S_{e\mu} &\sim -1 && \longleftrightarrow && S_{e\tau} && \sim +1 \\ S_{\mu\mu} &\sim -2 && \longleftrightarrow && S_{\tau\tau} && \sim +2\,. \end{aligned} \tag{II-3.60}$$

We see that there are always two scalars with either the same or conjugate charged, meaning that we also have to add the terms with conjugate fields to the Lagrangian! I.e., assuming S_{ee} and $S_{\mu\tau}$ are real, we also need the terms

$$-2\mathcal{L} \supset \tilde{\lambda}_{ee}\, S_{\mu\tau}\, \overline{N_1^C}\, N_1 + \tilde{\lambda}_{\mu\tau}\, S_{ee}\, \overline{N_2^C}\, N_3 + \tilde{\lambda}_{e\mu}\, S^\dagger_{e\tau}\, \overline{N_1^C}\, N_2 + \tilde{\lambda}_{e\tau}\, S^\dagger_{e\mu}\, \overline{N_1^C}\, N_3 + \\ + \tilde{\lambda}_{\mu\mu}\, S^\dagger_{\tau\tau}\, \overline{N_2^C}\, N_2 + \tilde{\lambda}_{\tau\tau}\, S^\dagger_{\mu\mu}\, \overline{N_3^C}\, N_3 + \text{h.c.}\,. \tag{II-3.61}$$

After breaking $U(1)_{L_\mu - L_\tau}$, and assuming real vacuum expectation values (vevs), this would yield a mass matrix

$$M_R = \begin{pmatrix} M_{ee} + \lambda_{ee}\,\langle S_{ee}\rangle & \lambda_{e\mu}\,\langle S_{e\mu}\rangle & \lambda_{e\tau}\,\langle S_{e\tau}\rangle \\ \lambda_{e\mu}\,\langle S_{e\mu}\rangle & \lambda_{\mu\mu}\,\langle S_{\mu\mu}\rangle & M_{\mu\tau} + \lambda_{\mu\tau}\,\langle S_{\mu\tau}\rangle \\ \lambda_{e\tau}\,\langle S_{e\tau}\rangle & M_{\mu\tau} + \lambda_{\mu\tau}\,\langle S_{\mu\tau}\rangle & \lambda_{\tau\tau}\,\langle S_{\tau\tau}\rangle \end{pmatrix} + \\ + \begin{pmatrix} \tilde{\lambda}_{ee}\,\langle S_{\mu\tau}\rangle & \tilde{\lambda}_{e\mu}\,\langle S_{e\tau}\rangle & \tilde{\lambda}_{e\tau}\,\langle S_{e\mu}\rangle \\ \tilde{\lambda}_{e\mu}\,\langle S_{e\tau}\rangle & \tilde{\lambda}_{\mu\mu}\,\langle S_{\tau\tau}\rangle & \tilde{\lambda}_{\mu\tau}\,\langle S_{e}\rangle \\ \tilde{\lambda}_{e\tau}\,\langle S_{e\mu}\rangle & \tilde{\lambda}_{\mu\tau}\,\langle S_{e}\rangle & \tilde{\lambda}_{\tau\tau}\,\langle S_{\mu\mu}\rangle \end{pmatrix} . \tag{II-3.62}$$

However, now all couplings always appear in pairs, so we cannot create too different entries, unless the respective Yukawa couplings are very different. Without these terms, we may assume Yukawa couplings of the same order of magnitude, but, e.g., different scales for the vevs to realize random entries. Furthermore, having these additional interaction terms would complicate calculations without great benefit. Therefore, we would like to drop the $\tilde{\lambda}_{ij}$ terms from the

Lagrangian.

One way, would be to only use three scalars instead of six, and give these the charges

$$S_0 \sim 0\,, \quad S_1 \sim -1\,, \quad S_2 \sim -2\,. \tag{II-3.63}$$

These would lead to a Lagrangian

$$\begin{aligned}-2\mathcal{L} \supset\, & Y_{\nu,i}\,\overline{l_i}\,\epsilon\,\phi^*\,N_i + \lambda_{ee}\,S_0\,\overline{N_1^C}\,N_1 + \lambda_{\mu\tau}\,S_0\,\overline{N_2^C}\,N_3 + \lambda_{e\mu}\,S_1\,\overline{N_1^C}\,N_2 + \lambda_{e\tau}\,S_1^\dagger\,\overline{N_1^C}\,N_3 + \\ & + \lambda_{\mu\mu}\,S_2\,\overline{N_2^C}\,N_2 + \lambda_{\tau\tau}\,S_2^\dagger\,\overline{N_3^C}\,N_3 + \frac{1}{2}\,M_e e\,\overline{N_1^C}\,N_1 + M_{\mu\tau}\,\overline{N_2^C}\,N_3 + \text{h.c.}\,,\end{aligned} \tag{II-3.64}$$

where we see that while the conjugate scalars appear, there is only one coupling of every type. This would thus lead us to a mass matrix as in the first line of eq. (II-3.62), with the substitutions $\langle S_{ee}\rangle$, $\langle S_{\mu\tau}\rangle \longrightarrow \langle S_0\rangle$; $\langle S_{e\mu}\rangle$, $\langle S_{e\tau}\rangle \longrightarrow \langle S_1\rangle$; and $\langle S_{\mu\mu}\rangle$, $\langle S_{\tau\tau}\rangle \longrightarrow \langle S_2\rangle$. So once again, the entries are coupled by the vevs.

Nevertheless, let us for comparison with the six-scalar model consider the mass Z' would incur from the scalar vevs. The Z' gains a mass via the Higgs mechanism from the gauge covariant derivatives of the scalars,

$$\mathcal{L}_{kin} \supset \left(D^\mu S_0\right)^\dagger \left(D_\mu S_0\right) + \left(D^\mu S_1\right)^\dagger \left(D_\mu S_1\right) + \left(D^\mu S_2\right)^\dagger \left(D_\mu S_2\right), \tag{II-3.65}$$

$$D_\mu^{(S_0,\,S_1,\,S_2)} = \partial_\mu + i\,\tilde{g}\,(0,\,-1,\,-2)\,Z'_\mu\,. \tag{II-3.66}$$

After symmetry breaking, this contains the mass terms for the Z', and we find

$$\begin{aligned}\mathcal{L} &\supset \tilde{g}\left(0\cdot\langle S_0\rangle^2 + (-1)^2\cdot\langle S_1\rangle^2 + (-2)^2\cdot\langle S_2\rangle^2\right) Z'^\mu\,Z'_\mu \\ &= \frac{1}{2}\,\tilde{g}\left(2\cdot\langle S_1\rangle^2 + 2\cdot 4\cdot\langle S_2\rangle^2\right) Z'^\mu\,Z'_\mu\end{aligned} \tag{II-3.67}$$

$$\Longrightarrow \quad M_{Z'} = \sqrt{2}\,\tilde{g}\cdot\sqrt{\langle S_1\rangle^2 + 4\langle S_2\rangle^2}\,. \tag{II-3.68}$$

This is a relatively compact formula, and lends itself well to discussing some general properties. Namely, as discussed in the previous chapter, to get renormalization effects from the Z', it needs to be lighter than the renormalization scale, as well as the UV scale Λ. However, the right-handed neutrinos obtain their masses from the symmetry breaking vevs—we neglect the ones that are symmetry-allowed for now. In a Type-I Seesaw, the scale Λ up to which the Weinberg-Operator offers a reasonable description of neutrino masses, corresponds to the right-handed neutrino mass scale. Recall that since the intermediate, right-handed neutrinos are integrated out, their propagator is basically given by their inverse mass, and thus $\kappa \sim 1/\Lambda \sim 1/M_N$. However, we also know that $M_N \sim \langle S\rangle$, and that $M_{Z'} \sim \langle S\rangle$. Therefore, to obtain sizable RGE effects coming from the Z', we need to assume at least, e.g., one order of magnitude difference of the mass of the lightest right-handed neutrino, and Z'. This means that the couplings $\tilde{g}$ and

λ_{ij} play an important role in determining whether the new effects discussed in this work occur. Note that large values of the flavor-neutral vevs can increase the likelihood of this happening, as the Z' mass does not increase, while the right-handed neutrinos' masses do. However, this would also yet again lead to a noticeable $L_\mu - {}_\tau$ symmetry structure.

While also the three-scalar model is interesting, we will not consider it further here, as the parametric freedom of the six-scalar model is more appealing to realize structure-free entries of κ at the scale Λ.

Let us now return to the six-scalar model. To be able to drop the $\tilde{\lambda}$ terms, we may like to enforce some additional global symmetry. Consider the terms

$$-\mathcal{L} \supset Y_{\nu,i}\,\overline{l_i}\,\epsilon\,\phi^*\,N_i + \frac{1}{2}\lambda_{ij}\,S_{ij}\,\overline{N_i^C}\,N_j \;+ \text{h.c.}\,. \tag{II-3.69}$$

We would like them to be invariant under a global $U(1)$ symmetry, such that the $\tilde{\lambda}$ terms are forbidden due to the appearing $S_{ij}^\dagger$. To keep both terms in eq. (II-3.69) invariant, the transformation has to be of the form

$$N_i \longrightarrow e^{i\,\alpha}\,N_i\,, \quad l_i \longrightarrow e^{i\,\alpha}\,l_i\,, \quad S_{ij} \longrightarrow e^{-2i\,\alpha}\,. \tag{II-3.70}$$

We notice that what we have discovered here is just lepton-number! In fact, this is precisely the transformation behavior proposed in so-called majoron models—models with spontaneously broken lepton number, where the emerging Nambu-Goldstone bosons are called majorons; see, e.g., [53]. Since the scalars obtain a vev, they do not only break the gauged $U(1)_{L_\mu - L_\tau}$ spontaneously, but also the global lepton number symmetry. However, there are problems associated with such spontaneous breaking of lepton number, for instance, the cosmological domain wall problem [54]. The origin of this emerging problem is that the residual symmetry left unbroken by the scalars obtaining vevs is mismatched with the residual symmetry left unbroken by instanton effects in the electroweak sector. Due to the mismatch of the vacuum state's symmetry groups, domain walls may appear. However, these are cosmologically not viable and must thus be eliminated—see [54] and references therein. One proposed way of dealing with this problem is to introduce an $SU(3)$ lepton flavor gauge group, whereby the difference between the vacua is made unphysical, as they are related by gauge transformations. This motivates a model for an $SU(3)$ flavor gauge theory, and considering its effects on the renormalization of κ. However, as it would go beyond the scope of this thesis, we do not discuss it further here. Note that [54] also proposes alternative solutions to the problem, whereby new fermion multiplets resolve the issue, but we will not comment on this further.

Therefore, instead of explicitly imposing a global symmetry to eliminate the $\tilde{\lambda}$ terms, we assume some mechanism whereby they drop out, and neglect them henceforth.

II-3.4.1 RGEs for the Right-Handed Neutrinos

Let us now calculate the β-function for the right-handed neutrinos in this extended model. In the Standard Model, the RGEs for the right-handed neutrinos were calculated in [15] at the

one-loop, and in [55] at the two-loop level. Note that we restrict ourselves to the one-loop level here.
Note that we will preform the renormalization before SSB because it is significantly easier to do in the symmetric phase, and the results should still describe the running after SSB. The reason for this is that, e.g., writing the scalars as excitations around the vacuum expectation value corresponds to reparametrizing the theory, which should not affect the results.

To obtain the masses for the right-handed neutrinos after SSB at the one-loop level, we need to renormalize both the tree-level mass matrix, as well as the Yukawa couplings. However, we thus first need to consider the kinetic terms of the S_{ij} scalars, and their coupling to Z'. We define the gauge covariant derivative acting on the scalar S_{ij} as

$$D_\mu = \partial_\mu + i\,\tilde{g}\,Y^S_{ij}\,Z'_\mu\,, \tag{II-3.71}$$

where Y^S_{ij} is the flavor charge of S_{ij}. The Y^S_{ij} may also be written as a matrix of the form

$$Y^S = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\,. \tag{II-3.72}$$

In general, we may couple kinetic terms of the scalars in the form of, e.g., $\left[D^\mu S_{12}\right] D_\mu S_{13}$, as they are still gauge-invariant by virtue of the opposing gauge charges. However, this complicates the model and calculations therein significantly. Therefore, we assume these mixed kinetic terms to vanish and will neglect them for simplicity. Thus, summing over pairs of ij, the kinetic terms become

$$\begin{aligned}\mathcal{L}_{kin,\,S} = \left[D^\mu S_{ij}\right]^\dagger D_\mu S_{ij} = &\left|\partial_\mu S_{ij}\right|^2 + i\,\tilde{g}\,Y^S_{ij}\,Z'_\mu\,S_{ij}\,\partial^\mu\,S^\dagger_{ij} - i\,\tilde{g}\,Y^S_{ij}\,Z'_\mu\,S^\dagger_{ij}\,\partial^\mu S_{ij} + \\ &+\ \tilde{g}^2\left(Y^S_{ij}\right)^2 Z'_\mu\,Z'^\mu\,S^\dagger_{ij}\,S_{ij}\,.\end{aligned} \tag{II-3.73}$$

Note that we take S_{ee} and $S_{\mu\tau}$ to be complex scalars. As usual, we obtain cubic and quartic interactions with the gauge boson. Note that in our considerations, the quartic interaction does not play a role.

First, we renormalize the tree-level mass matrix to obtain its β-function, β_M. In particular, we consider it as a two-point interaction vertex between massless, right-handed neutrinos. Since we are at the one-loop level, we follow an analogous line of arguments as for κ, noting that $D_M = 0$. We thus obtain that, at the one-loop level, the β_M-function is given by

$$\beta_M = \delta M_{,1} - \frac{1}{2}\,\delta Z^T_N\,M - \frac{1}{2}\,M\,\delta Z_N\,. \tag{II-3.74}$$

Note that we define the bare coupling by

$$M_B = \left(Z^T_M\right)^{-1/2}\left[M + \delta M\right] Z^{-1/2}_M\,, \tag{II-3.75}$$

such that we can proceed analogously as for $\delta\kappa$. Thus, we need to consider the contributions to the wave function renormalization of N_i, and the vertex corrections. In the SM, the only contributions come from wave function renormalization, and the β_M-function is given by [15]

$$16\pi^2\,\beta_M = \left(Y_\nu^\dagger\,Y_\nu\right)^T M + M\left(Y_\nu^\dagger\,Y_\nu\right). \tag{II-3.76}$$

This contribution comes from the diagram shown in fig. (II-3.3), i.e., a self-energy bubble with the lepton- and Higgs-doublet as intermediate particles.

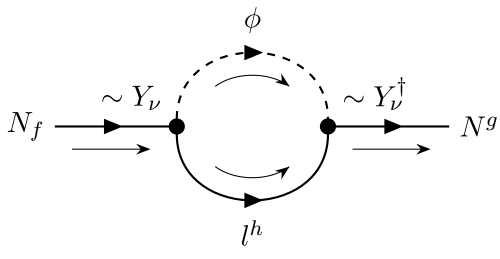


Figure II-3.3.: Wave function renormalization of the right-handed neutrinos coming from the Yukawa coupling to the left-handed lepton-doublet, and the Higgs-doublet in the Standard Model

In our extended model, we now also have contributions to wave function renormalization coming from the new scalars and the Z'. Furthermore, the Z' also contributes to vertex renormalization. The scalars cannot contribute to vertex renormalization because they are charged, and would thus need opposite flow on one of the external legs to be able to couple on both sides of M; however, this is incompatible with the flow required for M itself, and thus the scalars cannot contribute. In fact, we may also neglect the scalars' contribution to wave function renormalization because their coupling to N will induce masses after SSB, such that the contribution to κ will be of order $1/\Lambda^3$—one power from M, two from the Yukawa coupling λ. Therefore, we are left only with the Z' contributions to wave function and vertex renormalization.

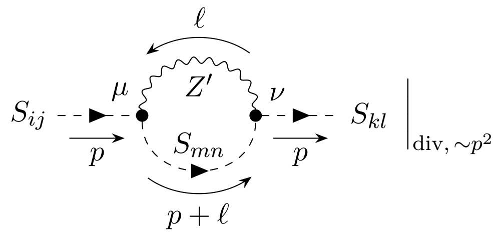
To calculate these contributions, we can directly employ eq. (II-2.297) and (II-2.271). Even though they were derived for κ and the lepton-doublet, differing signs, the $SU(2)_L$ structure, projectors etc. drop out since the counterterms exhibit them as well. Furthermore, since the integral structure is the same for the contributions renormalizing M as for κ, we may hence use the equations here as well. Thus, we can directly obtain the β_M-function to be

$$\begin{aligned}
16\pi^2\,\beta_M =& 16\pi^2\left(\beta_{M,SM} + \beta_{M,U(1)_{L_\mu - L_\tau}}\right) && \text{(II-3.77)}\\
=& \left[\tilde{g}^2\left(\tilde{Q}\right)^2\tilde{\xi} + \left(Y_\nu^\dagger\,Y_\nu\right)\right]^T M + M\left[\tilde{g}^2\left(\tilde{Q}\right)^2\tilde{\xi} + \left(Y_\nu^\dagger\,Y_\nu\right)\right] + && \text{(II-3.78)}\\
&+\ 2\tilde{g}^2\,(3+\tilde{\xi})\left(\tilde{Q}^T\,M\,\tilde{Q}\right)\\
=& \left(Y_\nu^\dagger\,Y_\nu\right)^T M + M\left(Y_\nu^\dagger\,Y_\nu\right) + 6\,\tilde{g}^2\left(\tilde{Q}^T\,M\,\tilde{Q}\right). && \text{(II-3.79)}
\end{aligned}$$

Note that while it at first seems like the β_M-function is gauge-dependent, it is indeed gauge-invariant when checking component-wise, and recalling that only M_{ee} and M_{23} are nonzero.

Therefore, we simply dropped the terms proportional to $\tilde{\xi}$ in the last line. Furthermore, even though a $\tilde{Q}^T M \tilde{Q}$ term is present, we note that it cannot raise the rank of M, due to only the M_{ee} and $M_{\mu\tau}$ entries being different from zero. Thus, the incurred signs in the $\mu\tau$ entries compared to the here vanishing $\mu\mu$ and $\tau\tau$ entries do not affect the evolution of M.

Next, we consider the renormalization of λ_{ij}. Following similar arguments as before, the only contributions come from Z'. Note that we neglect possible cubic scalar interactions, as these do not impact flavor-space structure if we assume all possible cubic couplings to be of the same order. For the wave function renormalization of N_i and the vertex correction coming from the Z', we can once again employ eq. (II-2.297) and (II-2.271). However, we also need to consider the Z' contribution from coupling to the S_{ij}. The wave function renormalization is given by



$$= \qquad \text{(II-3.80)}$$

$$= \int \frac{\mathrm{d}^d\ell}{(2\pi)^d} \Big[-i\,\tilde{\mu}^{\varepsilon}\,\tilde{g}\,Y^S_{kl}\,\delta_{mn,\,kl}\,\big(p+\ell+p\big)_\nu \Big] \frac{i}{(p+\ell)^2 - m^2_{ij}} \quad \times \tag{II-3.81}$$

$$\times \quad \Big[-i\,\tilde{\mu}^{\varepsilon}\,\tilde{g}\,Y^S_{ij}\,\delta_{ij,\,mn}\,\big(p+p+\ell\big)_\nu \Big] \frac{i}{\ell^2} \left[-g^{\mu\nu} + (1-\tilde{\xi})\,\frac{\ell^\mu\,\ell^\nu}{\ell^2} \right] \Bigg|_{\mathrm{div},\,\sim p^2}$$

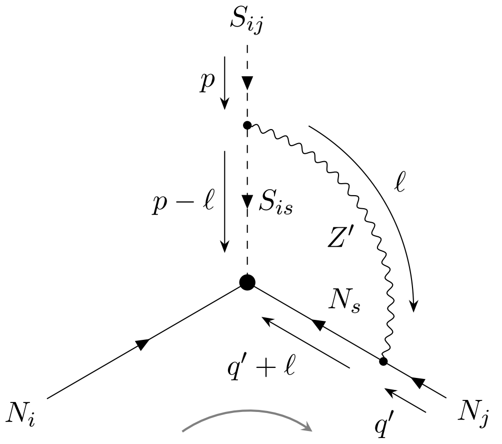
$$= -\big(-i\,\tilde{g}\,Y^S_{ij}\big)^2\,\delta_{ij,\,kl} \cdot \int \big[\mathrm{d}^d\ell\big]\,\big(2p+\ell\big)_\mu\,\big(2p+\ell\big)_\nu\,\frac{1}{(p+\ell)^2 - m^2_{ij}} \quad \times \tag{II-3.82}$$

$$\times \quad \frac{1}{\ell^2} \left[-g^{\mu\nu} + (1-\tilde{\xi})\,\frac{\ell^\mu\,\ell^\nu}{\ell^2} \right] \Bigg|_{\mathrm{div},\,\sim p^2}$$

$$= -\big(-i\,\tilde{g}\,Y^S_{ij}\big)^2\,\delta_{ij,\,kl} \cdot \frac{i}{16\pi^2}\,(3-\tilde{\xi})\,\frac{-1}{\varepsilon} \tag{II-3.83}$$

$$= -\frac{i}{16\pi^2}\,p^2\,\delta_{ij,\,kl}\,\frac{1}{\varepsilon}\,\tilde{g}^2\,(3-\tilde{\xi})\,\big(Y^S_{ij}\big)^2. \tag{II-3.84}$$

The new contribution to vertex renormalization is given by



$$\Big|_{\text{div},\,\sim p^2} \quad + \quad \text{conjugate diagram} \quad = \tag{II-3.85}$$

$$= \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \Big[-i\,\tilde{\mu}^{\varepsilon}\,\tilde{g}\,\tilde{Q}_{sj}\,\big(-\gamma_\nu\,P_L\big)\Big]\,\frac{i\big(-\not{q}'-\not{\ell}\big)}{(q'+\ell)^2}\,\big[-i\,\lambda_{si}\,P_R\big]\,\frac{i}{(p-\ell)^2-m_{ij}^2} \quad \times \tag{II-3.86}$$

$$\times \quad \Big[-i\,\tilde{\mu}^{\varepsilon}\,\tilde{g}\,Y^S_{is}\,\delta_{ij,\,is}\,\big(p+p-\ell\big)_\mu\Big]\,\frac{i}{\ell^2}\left[-g^{\mu\nu}+(1-\tilde{\xi})\,\frac{\ell^\mu\,\ell^\nu}{\ell^2}\right]\Bigg|_{\text{div}} + \text{ c. d.}$$

$$= -\,(-i\,\tilde{g})\,\big(-i\,\tilde{g}\,Y^S_{ij}\big)\,\big(\tilde{Q}^T\,\lambda\big)_{ji}\,P_R\cdot\int\big[\mathrm{d}^d\ell\big]\,\gamma_\nu\,\frac{\not{q}'+\not{\ell}}{(q'+\ell)^2}\,\frac{1}{(p+\ell)^2-m_{ij}^2} \quad \times \tag{II-3.87}$$

$$\times \quad \big(p+p-\ell\big)_\mu\,\frac{1}{\ell^2}\left[-g^{\mu\nu}+(1-\tilde{\xi})\,\frac{\ell^\mu\,\ell^\nu}{\ell^2}\right]\Bigg|_{\text{div}} + \text{ c. d.}$$

$$= -\,(-i\,\tilde{g})\,\big(-i\,\tilde{g}\,Y^S_{ij}\big)\,\big(\tilde{Q}^T\,\lambda\big)_{ji}\,P_R\cdot\frac{i}{16\pi^2}\,\tilde{\xi}\,\frac{1}{\varepsilon} + \text{ c. d.} \tag{II-3.88}$$

$$= \frac{i}{16\pi^2}\,\frac{1}{\varepsilon}\,P_R\,\tilde{g}^2\,\tilde{\xi}\,Y^S_{ij}\,\big(\tilde{Q}^T\,\lambda+\lambda\,\tilde{Q}\big)_{ji}. \tag{II-3.89}$$

Thus, we now have all the contributions necessary to for the β_λ-function. Note again that tadpole diagrams do not contribute to wave function renormalization and can thus be neglected; the same is true for the contributions $\sim m_{ij}^2$ from bubble diagrams. Let us also re-emphasize that it is crucial to use an IR regulator when calculating the renormalization constants. In practice, this can be done, e.g., by putting a mass for the Z' and setting it to zero at the end. By now canceling the divergences via their respective counterterms, we obtain the renormalization constants

$$\delta Z_{S_{ij},1} = \frac{-2}{16\pi^2}\left(-(3-\tilde{\xi})\right)\tilde{g}^2\,(Y^S_{ij})^2\,, \tag{II-3.90}$$

$$\delta\lambda_{ji} = \frac{1}{16\pi^2}\left[2\tilde{g}^2\,(3+\tilde{\xi})\left(\tilde{Q}^T\,\lambda\,\tilde{Q}\right)_{ji} + 2\tilde{g}^2\,\tilde{\xi}\,Y^S_{ij}\,\left\{\tilde{Q}^T\,\lambda + \lambda\,\tilde{Q}\right\}_{ji}\right]. \tag{II-3.91}$$

Note that the Feynman-rules for the counterterms are listed in the appendix. Finally, we use eq. (I-3.59) and find for the β_λ-function,

$$\begin{aligned}
16\pi^2\,\beta^{ji}_\lambda =& 16\pi^2\left[\delta\lambda_{ji,1} - \frac{1}{2}\,\delta Z_{S_{ij},1}\,\lambda_{ji} - \frac{1}{2}\left(\delta Z^T_{N,1}\,\lambda\right)_{ji} - \frac{1}{2}\left(\lambda\,\delta Z_{N,1}\right)_{ji}\right] && \text{(II-3.92)}\\
=& -(3-\tilde{\xi})\,\tilde{g}^2\left(Y^S_{ij}\right)^2\lambda_{ji} + \lambda_{jk}\left[\tilde{g}^2\,\tilde{\xi}\,(\tilde{Q})^2 + 2\tilde{g}^2\,\tilde{\xi}\,Y^S_{ij}\,\tilde{Q} + \left(Y^\dagger_\nu\,Y_\nu\right)\right]_{ki} + && \text{(II-3.93)}\\
&+ \left[\tilde{g}^2\,\tilde{\xi}\,(\tilde{Q})^2 + 2\tilde{g}^2\,\tilde{\xi}\,Y^S_{ij}\,\tilde{Q} + \left(Y^\dagger_\nu\,Y_\nu\right)\right]^T_{jk}\lambda_{ki} + \\
&+ 2\tilde{g}^2\,(3+\tilde{\xi})\left(\tilde{Q}^T\,\lambda\,\tilde{Q}\right)_{ji}\\
=& -3\tilde{g}^2\left(Y^S_{ij}\right)^2\lambda_{ji} + \left[\left(Y^\dagger_\nu\,Y_\nu\right)^T\lambda\right]_{ji} + \left[\lambda\left(Y^\dagger_\nu\,Y_\nu\right)\right]_{ji} + 6\tilde{g}^2\left(\tilde{Q}^T\,\lambda\,\tilde{Q}\right)_{ji}. && \text{(II-3.94)}
\end{aligned}$$

Here, we again verified explicitly that the gauge parameter cancels in every component, such that we can drop the terms proportional to it. Furthermore, we found that the term $\sim \left(Y^S_{ij}\right)^2\lambda_{ji}$ can be rewritten in terms of $\tilde{Q}$:

$$\left(Y^S_{ij}\right)^2\lambda_{ji} = \left[\tilde{Q}^2\,\lambda + \lambda\,\tilde{Q}^2 + 2\tilde{Q}^T\,\lambda\,\tilde{Q}\right]_{ji}. \tag{II-3.95}$$

The underlying reason for this is that the charges of S_{ij} are defined exactly such that they cancel the charges of N_i in the interaction terms, and are thus implicitly defined by $\tilde{Q}$. Thus, we find that the $\tilde{Q}^T\,\lambda\,\tilde{Q}$ term in fact cancels. The final β_λ-function we are left with is given by

$$16\pi^2\,\beta_\lambda = \left[-3\tilde{g}^2\,\tilde{Q}^2 + Y^\dagger_\nu\,Y_\nu\right]^T\lambda + \lambda\left[-3\tilde{g}^2\,\tilde{Q}^2 + Y^\dagger_\nu\,Y_\nu\right]. \tag{II-3.96}$$

After SSB, the RGEs for the total mass matrix for the right-handed neutrinos is therefore given by

$$\beta^{ji}_{M_R} = \beta^{ji}_{M} + \langle S_{ij}\rangle\, \beta^{ji}_{\lambda} \tag{II-3.97}$$

$$= \underbrace{6\tilde{g}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -M_{\mu\tau} \\ 0 & -M_{\mu\tau} & 0 \end{pmatrix}}_{16\pi^2\, \beta^{ji}_M} \underbrace{-3\tilde{g}^2 \begin{pmatrix} 0 & \langle S_{e\mu}\rangle\, \lambda_{e\mu} & \langle S_{e\tau}\rangle\, \lambda_{e\tau} \\ \langle S_{e\mu}\rangle\, \lambda_{e\mu} & 2\langle S_{\mu\mu}\rangle\, \lambda_{\mu\mu} & 2\langle S_{\mu\tau}\rangle\, \lambda_{\mu\tau} \\ \langle S_{e\tau}\rangle\, \lambda_{e\tau} & 2\langle S_{\mu\tau}\rangle\, \lambda_{\mu\tau} & 2\langle S_{\tau\tau}\rangle\, \lambda_{\tau\tau} \end{pmatrix}}_{16\pi^2 \langle S_{ij}\rangle\, \beta^{ji}_{\lambda}} \tag{II-3.98}$$

$$+ \text{Yukawa terms} \tag{II-3.99}$$

$$= \Big[-3\tilde{g}^2\, \tilde{Q}^2 + Y_\nu^\dagger\, Y_\nu \Big]^T M_R + M_R \Big[-3\tilde{g}^2\, \tilde{Q}^2 + Y_\nu^\dagger\, Y_\nu \Big]\,, \tag{II-3.100}$$

where we identified the total, tree-level mass matrix M_R after SSB in the last step.

Thus, we see that for the right-handed neutrinos, the Z' interaction cannot, after all, increase the rank of M_R—the various contributions end up canceling the $G^T\, M_R\, G$ term. However, this is only valid as long as all scalars are active. If some scalars become are heavier than the renormalization scale, the become inactive and do not participate in the renormalization of M_R anymore. Thus, in such scenarios, the contributions coming from the Z' coupling to this scalar would vanish, allowing us to regain a $G^T\, M_R\, G$ term in the β_{M_R}-function. This is analogous to what we discussed earlier concerning $SU(N)$ flavor gauge theories in the fundamental representation.

Thus, we have obtained the final result for the β-function of the right-handed neutrino mass matrix. In particular, while we expressed the flavor-space matrices explicitly for $U(1)_{L_\mu - L_\tau}$ in some cases for clarity, our findings here extend to a general $\tilde{Q}$ in any gauged $U(1)$ flavor symmetry. That is, by taking the results in matrix notation and plugging in the specific structure in a given $U(1)$ theory, the results can be applied to any such flavor gauge extension. Let us now briefly summarize our findings. Note that we also add the mass the Z' obtains after SSB. We obtain this analogously to eq. (II-3.68) in the three-scalar model from the vevs of the flavor-charged scalars.

We have introduced a model with **six scalars** S_{ij} whose subscripts refer to the right-handed neutrino flavors they couple to. In a **general $U(1)$ flavor gauge theory** with a traceless charge matrix, we choose the charges such that they compensate for the repsective neutrino charges. The Lagrangian giving rise to right-handed neutrino masses is given by

$$-\mathcal{L} \supset Y_{\nu,i}\,\overline{l_i}\,\epsilon\,\phi^*\,N_i + \frac{1}{2}\,\lambda_{ij}\,S_{ij}\,\overline{N_i^C}\,N_j + \frac{1}{2}\,M_{ij}\,\overline{N_i^C}\,N_j\, + \text{h.c.}\,, \qquad \text{(II-3.101)}$$

where M_{ij} has a flavor $U(1)$-symmetric structure. The resulting **tree-level Majorana mass matrix after Spontaneous Symmetry Breaking**, $\boldsymbol{M_R}$ and $\boldsymbol{\beta_{M_R}}$**-function** are given by

$$M_R^{ij} \;=\; M_{ij} \;+\; \langle S_{ij}\rangle\lambda_{ij}\,, \qquad \text{(II-3.102)}$$

$$\begin{aligned} 16\pi^2\,\beta_{M_R} \;=& \Big[-\,3\tilde{g}^2\,\tilde{Q}^2 \;+\; Y_\nu^\dagger\,Y_\nu \Big]^T \;M_R\; + \\ &+\; M_R\,\Big[-\,3\tilde{g}^2\,\tilde{Q}^2 \;+\; Y_\nu^\dagger\,Y_\nu \Big]\,. \end{aligned} \qquad \text{(II-3.103)}$$

While the **rank of the mass matrix cannot be raised in the full theory**, it **may be raised after integrating out some of the heavy scalars**.

The contribution from the Yukawa coupling to the Higgs-doublet is taken from [15].

The mass of the Z' after SSB is given by

$$M_{Z'} \;=\; \sqrt{2}\,\tilde{g}\,\cdot\,\sqrt{\sum_{(ij)} \big(Y_{ij}^S\big)^2\,\langle S_{ij}\rangle^2} \qquad \text{(II-3.104)}$$

$$\overset{U(1)_{L_\mu - L_\tau}}{=} \quad \sqrt{2}\,\tilde{g}\,\sqrt{\langle S_{e\mu}\rangle^2 + \langle S_{e\tau}\rangle^2 + 4\langle S_{\mu\mu}\rangle^2 + 4\langle S_{\tau\tau}\rangle^2} \qquad \text{(II-3.105)}$$

II-3.4.2 The β_κ-Function in the Six-Scalar $U(1)$ Extension

After having discussed the RGEs of the right-handed neutrino mass matrix in the six-scalar $U(1)$ gauge extension, we will now discuss the β_κ-function in this model.

After SSB, we can parametrize the scalars as

$$S_{ij} = \frac{v_{ij} + \sigma_{ij} + i\,\rho_{ij}}{\sqrt{2}}\,, \tag{II-3.106}$$

where the vacuum expectation value is given by $\langle S_{ij}\rangle = v_{ij}/\sqrt{2}$, σ_{ij} are the massive excitations and ρ_{ij} the massless ones. This is a usual decomposition for complex scalar fields after they obtain a vev. Note that we can see that σ_{ij} obtain masses given by the quadratic coupling of S_{ij}, while ρ_{ij} remain massless by inspecting the scalar potential. Since this is a standard result, we will not show it here—see, e.g., [36]. Let us comment, however, on the scale of the masses. The vacuum expectation values are generically obtained from the minima of

$$-\mathcal{L}_{\text{scalar pot.}} \supset -\mu_{ij}^2 \big(S_{ij}^\dagger\, S_{ij}\big) + \frac{\lambda_{ij}^S}{2}\,\big(S_{ij}^\dagger\, S_{ij}\big)^2\,, \tag{II-3.107}$$

and are given by

$$v_{ij} = \sqrt{\frac{\mu_{ij}^2}{\lambda_{ij}^S}}\,. \tag{II-3.108}$$

Thus, the masses of the right-handed neutrinos roughly correspond to

$$M_R \sim \lambda_{ij}\sqrt{\frac{\mu_{ij}^2}{\lambda_{ij}^S}}\,. \tag{II-3.109}$$

However, the emerging masses of the σ_{ij} are given by

$$m_{\sigma_{ij}} \sim \mu_{ij}\,. \tag{II-3.110}$$

Therefore, the remaining massive scalars and right-handed neutrinos are roughly of the same mass scale. This means that in the effective theory, when we integrate out the right-handed neutrinos, we also need to integrate out the heavy scalars. Therefore, these do not participate in the renormalization of κ. We may still have, in principle, contributions coming from the massless ρ_{ij}. Nevertheless, we do not need to consider these at the one-loop level. Namely, since they only couple to the right-handed neutrinos and the Z', we would at least need a two-loop process to involve the ρ_{ij} in the effective theory. Note that this includes the gauge-dependent Goldstone mode. Thus, the β_κ-function we computed in eq. (II-3.15) without considering scalar contributions, holds even in our extended model. In this case, the entries of κ are given by the seesaw formula of eq (I-2.60),

$$\kappa = -2\,Y_\nu^*\, M_R^{-1}\, Y_\nu^\dagger\,. \tag{II-3.111}$$

II-3.4.3 Gauge-Noninvariance of the β_κ-Function

Inspecting eq. (II-3.15), we notice that the gauge parameter appears explicitly. If only the κ_{11} and κ_{23} entries are nonzero, all dependence of $\tilde{\xi}$ cancels, and we obtain a gauge-independent β_κ-function. However, if any of the other entries are different from zero, this is not the case anymore, which is alarming, given that the physical β_κ-function is supposed to be gauge-invariant. Let us therefore discuss this aspect and its origin. Note that this does not only apply to the $U(1)$ case, but is representative for other models as well.

To understand the origin of this gauge-dependence, let us go through the paradigm we work in. We consider a framework wherein we have a flavor gauge symmetry, which we then break spontaneously to obtain masses for the right-handed neutrinos. These masses come in the form of entries in the right-handed neutrino mass matrix that are not gauge-invariant. We then integrate out the right-handed neutrinos to obtain the Weinberg-Operator. However, κ thus obtains entries that are by definition not gauge-invariant. When we then renormalize κ and calculate the β_κ-function, we thus obtain a gauge-dependent object. Therefore, in summary, the gauge-dependence comes from the fact that we renormalize an effective operator via the gauge boson in the broken phase of the gauge theory. In fact, the effective operator is only defined in the broken phase, as the right-handed neutrinos obtain their masses from the spontaneous breaking of said symmetry. This makes the renormalization procedure inherently problematic. Usually, we require the β-functions of gauge-invariant operators to the gauge-invariant. However, in this case, the Weinberg-Operator is not gauge-invariant to begin with. And in fact, we find that the gauge-noninvariance of the β_κ-function depends directly on the gauge-noninvariance of the respective entries:

$$\beta_\kappa \Big|_{\propto\tilde{\xi}} = \frac{1}{16\pi^2}\,\tilde{g}^2\,\tilde{\xi}\begin{pmatrix} 0 & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & 4\kappa_{22} & 0 \\ \kappa_{13} & 0 & 4\kappa_{33} \end{pmatrix} \tag{II-3.112}$$

$$= \frac{1}{16\pi^2}\,\tilde{g}^2\,\tilde{\xi}\begin{pmatrix} (0+0)^2\,\kappa_{11} & (0+1)^2\,\kappa_{12} & (0-1)^2\,\kappa_{13} \\ (0+1)^2\,\kappa_{12} & (1+1)^2\,\kappa_{22} & (1-1)^2\,\kappa_{23} \\ (0-1)^2\,\kappa_{13} & (1-1)^2\,\kappa_{23} & (-1-1)^2\,\kappa_{33} \end{pmatrix}, \tag{II-3.113}$$

where we have made the charges of the respective fields entering the couplings explicit. These charges are then squared, since we couple to the Z' twice. We thus observe that the cause of the gauge dependence of β_κ, is the gauge-dependence of the respective interaction terms in the Weinberg-Operator.

Note that we cannot cancel the gauge-dependence of β_κ via couplings of the Z' to the Weinberg-Operator, as this would correspond to a dimension-six operator.

Despite understanding the origin of the seeming gauge-dependence, we need a way to deal with it in practice. One way to approach the issue would be to argue the following: Motivated by the observation that the gauge-dependence of the β_κ-function is directly related to the gauge-noninvariance of the respective entries, we propose that while there appears to be an apparent gauge-dependence, it must cancel implicitly. In other words, we require that the *total* derivative of the β_κ-function with respect to the gauge parameter vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{\xi}}\,\beta_{\mathcal{K}} \stackrel{!}{=} 0\,. \tag{II-3.114}$$

This does not forbid us from having an explicit dependence on $\tilde{\xi}$, as it must cancel via the gauge-noninvariance of κ. Fundamentally, this corresponds to assuming a non-manifest, but nevertheless present gauge-invariance. Building on this assumption of implicit cancelation of the gauge-dependencies, we then set $\tilde{\xi} \longrightarrow 0$, and drop all the gauge-dependent terms—this corresponds to choosing Landau gauge.

Second, inspired by [56], we compute the contributions of the Z' to the $\beta_{\mathcal{K}}$-function in unitary gauge, and compare the results to the ones obtained from the previously described prescription. And indeed, we find that when setting external momenta to zero, or respectively on-shell, our results in unitary gauge and Landau gauge agree. Setting external momenta to zero is justified, as we are considering the contributions to the $\beta_{\mathcal{K}}$, which cannot depend on external kinematics. Furthermore, we can assume the *external* particles to be on-shell. Since we thus find the same results as for Landau gauge, this further justifies the approach we described above.

Note that in future work, a more sophisticated approach to dealing with this problem should be employed. In [57], Background Field Gauge is employed to calculate β-functions in the spontaneously broken phase of the Standard Model. This formalism may provide a good framework also in our case, but implementing it here would go beyond the scope of this work. Therefore, we follow the approach outlined above, and set the gauge parameter to zero after calculating the contributions in a general R_ξ gauge.

Thus, we obtain the $\beta_{\mathcal{K}}$-function in the six-scalar model as

$$\beta_{\mathcal{K}} \;=\; bk_{SM} + \frac{6}{16\pi^2}\,\tilde{g}^2\left(\tilde{Q}^T\,\kappa\,\tilde{Q}\right) \tag{II-3.115}$$

We thus conclude the discussion of explicit models to implement flavor-nonuniversal renormalization of the Weinberg-Operator. While we also considered extensions to non-abelian models, we did not include these in this work, as it would have exceeded the scope of the thesis.

CHAPTER

II-4

Summary and Outlook

In this work, we considered radiative effects to the Weinberg-Operator in flavor-nonuniversally interacting gauge theories. In particular, we began our considerations by analyzing the effects of the new gauge boson arising in a $U(1)_{L_\mu - L_\tau}$ extension, on the β-function of the Weinberg-Operator, $\beta_{\mathcal{K}}$. We found on one hand that the contributions coming from wave function renormalization due to the new gauge boson, Z', fit within the same $\beta_{\mathcal{K}}$-function structure as the Standard Model (SM). On the other hand, we discovered a new type of structure that arises from vertex corrections due to the Z' coupling to both lepton-doublets. The new one-loop structure is of the form $G^T \kappa G$, where G is a matrix in flavor-space, and κ is the coupling matrix of the Weinberg-Operator. While such a structure appears at the two-loop level within the SM, it can already appear at the one-loop level in the presence of flavor-nonuniversally interacting vector bosons. It does not appear in the SM because the SM gauge group couples flavor-universally to all leptons, meaning that $G \propto \mathbb{1}$; therefore the $G^T \kappa G$ structure reduces to a scalar factor in front of κ.

After diagonalizing the neutrino mass matrix, we observed that this novel term is capable of raising the rank of the mass matrix at the one-loop level. This happens due to an emerging sum over all mass eigenvalues in the renormalization group equations (RGEs) for each of the eigenvalues. In the SM, the RGEs are proportional to the eigenvalues themselves, meaning that a vanishing eigenvalue cannot be generated via the RGEs. In the presence of the new term, however, we can radiatively induce neutrino masses simply by virtue of the renormalization group running. In particular, we compute the loop-corrections to the Weinberg-Operator at the high-energy (UV) scale that defines the effective mass operator, and then run them down to smaller scales. We have shown in an explicit example that starting with just one non-vanishing mass eigenvalue at the UV scale Λ, we can generate two further neutrino masses. Choosing a physically well-motivated numeric value for the initial condition of the nonzero mass yielded mass splittings that were very close to the experimental values. Therefore, given some initial conditions on the masses, the novel $G^T \kappa G$ structure offers a plausible explanation for both mass splittings, and mass hierarchies.

Furthermore, starting from the viewpoint of general symmetry requirements, loop-topological considerations, and scale arguments, we derived the most general renormalization group equation for the effective, dimension five neutrino mass operator at the one-loop level, and for any number of lepton generations. We found that in addition to the previously mentioned $G^T \kappa G$ term, a similar structure with two coupled flavor-space matrices is possible. This contribution to the $\beta_{\mathcal{K}}$-function is of the form $G_+^T \kappa G_- + G_-^T \kappa G_+$, and exhibits the same qualities as the $\sim G$ term. We also argued that without flow-reversing interactions for both the Higgs-doublet or the lepton-doublet, this structure holds to any loop level at the order $1/\Lambda$. We then derived the most general RGEs for the mass eigenvalues, and found that the $\sim G_\pm$ terms also produce a sum over all mass eigenvalues, meaning that they can lead to an enhanced running and mass generation. Furthermore, we derived the resulting RGEs for the leptonic mixing matrix.

We showed that at the one-loop level, $G_{(\pm)}$ cannot arise from lepton-number-violating interactions, and in particular, the only way to generate them is via vector boson interactions. Thus, vertex corrections where flavor-nonuniversally interacting vector bosons are exchanged between the lepton-doublets are the only possibility to induce the novel terms in the $\beta_{\mathcal{K}}$-function of the neutrino mass operator. Thus, if we wish to implement such contributions, it is necessary to consider flavor-nonuniversal gauge theories, and in particular flavor gauge theories. The only exception is given by independent, neutral, massive vector bosons, which are not required to be gauge bosons; however, flavor-nonuniversal coupling would have to be added in an ad-hoc manner, making this type of interaction theoretically less attractive. In particular, we showed that the only way to obtain $\sim G_\pm$ terms is via flavor-charged, non-abelian gauge bosons.

Additionally, we showed the necessary transformation behavior of the constituents that determine the possible flavor-space structure of the $\beta_{\mathcal{K}}$. We furthermore derived the most general decomposition of the constituents at the one-loop level, and showed the loop-topologies they can arise from. In particular, only triangle and bubble diagrams can contribute to the renormalization of κ. We then also showed that, in full generality, the scalar contribution is real and symmetric in any internal space, and the matrix constituents are hermitian. The exception to this are $G_\pm$, which are, however, the hermitian conjugates of each other. We understand this from the standpoint that they arise from coupling to flavor-charged gauge bosons and their antiparticles. Note that these findings hold in any extension of the SM.

While the previous analyses were done from the standpoint of the flavor-space structure, we also considered the general framework for in-practice calculation of the $\beta_{\mathcal{K}}$-function. We showed that the equation expressing the one-loop $\beta_{\mathcal{K}}$-function in terms of renormalization constants in the SM holds in any extension thereof, and can henceforth be applied universally. We then presented compact formulae to obtain the renormalization constants form divergent bubble and triangle diagrams. Furthermore, we calculated the general contributions from wave function renormalization and in particular the relevant vertex correction topology, which can now be used in a "plug and play" manner. This means that in any gauge theory we consider the Weinberg-Operator in, we can insert the corresponding flavor-space charge matrices or generators, and directly obtain the contribution to the $\beta_{\mathcal{K}}$-function. This makes it possible to check almost instantly whether a particular theory admits the new terms in the $\beta_{\mathcal{K}}$-function. Note that using this approach, we also showed that $SU(N)$ flavor gauge theories where the leptons are in the fundamental representation, and all gauge bosons participate in the renormalization of κ, do not lead to $\sim G_{(\pm)}$ terms. The individual terms cancel overall, and leave only a scalar contribution to β_{κ}. However, if some of the gauge bosons are integrated out and thus become inactive, the remaining ones will induce the new effects.

Lastly, we considered specific aspects of UV models to implement the novel RGE effects discovered in this work. In particular, we considered an abelian $U(1)_{L_\mu - L_\tau}$ extension of the Standard Model, where the symmetry is badly broken to allow for a structureless neutrino mass matrix at the UV scale. Therein, we derived the β_{κ}-function for any $U(1)$ flavor gauge extension, and provided an explicit formula to calculate the running entries of κ. Furthermore, we argued that to radiatively induce masses via the new $\sim G$ term, nonzero mixing angles at the UV scale Λ are required. We also solved the RGE for the mixing angle in the two-flavor case, and showed that it remains zero if it vanishes at Λ, and always increases otherwise.

Finally, we presented an explicit model wherein we introduced six new scalar fields, and three right-handed neutrinos in a Type-I seesaw mechanism. Since the model offers many tunable parameters, it allows us to assume a structureless neutrino mass matrix at the scale Λ. We then calculated the complete one-loop β-function for the right-handed neutrino mass matrix M_R in this model. We found that overall, $G^T M_R G$ terms either cancel, or do not raise the rank of M_R due to cancelation effects and symmetry constraints. Nevertheless, we showed that a $G^T \kappa G$ terms does emerge in the β_{κ}-function. However, we found that, since κ is only defined in the broken phase of the gauge symmetry, β_{κ} is seemingly gauge-dependent. This originates from the fact that the entries of κ generated via spontaneous symmetry breaking are by definition not gauge-invariant. Even so, we argued that the gauge-dependence, while not manifest, must cancel implicitly. Thus, we found that dropping the gauge-dependent terms in β_{κ}—corresponding to choosing Landau gauge—yields reasonable results. This also leads us to the outlook and further work to be done in the future.

As previously mentioned, gauge-invariance of the β_{κ}-function is not manifest at the one-loop level due to renormalization in the broken phase. While we found a reasonable way to deal with this, it is not satisfactory because it does not solve the underlying issue. Therefore, future work still needs to be done to renormalize the broken-phase, effective neutrino mass operator while maintaining manifest gauge-invariance of its β-function. For instance, background field gauge may offer one possible resolution to this issue.

Even though we did not present the results in this thesis, we also considered some aspects of specific non-abelian flavor gauge extensions to implement the new running effects. However, completing these to fully viable models is still necessary. For instance, future work should address how to arrange for the charged lepton masses, the specific symmetry breaking sequence of the gauge group, and potentially, integrating out of heavy gauge bosons. As previously mentioned, the last point is particularly important for $SU(N)$ flavor gauge groups in the fundamental representation—most notably $SU(3)$.

Lastly, an in-depth phenomenological analysis is still required. This means, for instance, examining the effects of particular parameter ranges of Yukawa and gauge couplings on the running of the neutrino masses. Furthermore, fixed points of mixing angles and phases still need to be considered, and whether the new contributions may drive these to particular fixed points.

Lastly, an analysis of potential experimental verification of the effects discussed in this work is needed. In particular, how to measure the high-energy effects considered here at accessible energy ranges. This includes a comparison of predictions arising from, e.g., the six-scalar model with experimental data, which is left to future work.

CHAPTER

A

General Addenda

In this appendix, we will supply additional information that may be useful for some aspects of the calculation, as well as the Feynman-rules we employed for calculations.

A.1 Useful Formulae

In this section, we will present useful formulae for the computation of Feynman diagrams and loop-integrals. See, for instance, [36].

A.1.1 Gamma-Matrices in d Dimensions

When computing Feynman diagrams by hand, the following identities for the gamma matrices prove very useful:

$$\gamma^\mu \gamma_\mu = d\,, \tag{A.1}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\,\gamma^\nu\,, \tag{A.2}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} - (4-d)\,\gamma^\nu \gamma^\rho\,, \tag{A.3}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\,\gamma^\nu \gamma^\rho\,, \gamma^\sigma\,. \tag{A.4}$$

Here, $g^{\mu\nu}$ is the Minkowski metric in d dimensions. Note that we use the mostly minus convention, where the metric signature is given by $(+,-,-,-)$. The relations above can be derived from

$$g^{\mu\nu}\, g_{\mu\nu} = d\,, \quad \text{and} \tag{A.5}$$

$$\left\{\gamma^\mu\,, \gamma^\nu\right\} = 2g^{\mu\nu}\,. \tag{A.6}$$

A.1.2 Feynman Parameters

A product of n propagators D_i with powers a_i, where $i \in [1, 2, ..., n]$, can be rewritten using Feynman parameters as

$$\frac{1}{D_1^{a_1}\, D_2^{a_2} \cdots D_n^{a_n}} = \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_n\, \delta\Big(\sum_i x_i - 1\Big)\, \frac{\prod_i x_i^{a_i - 1}}{\big[\sum_i x_i\, D_i\big]^{\sum_i a_i}} \times \\ \times\, \frac{\Gamma\big(a_1 + a_2 + \cdots + a_n\big)}{\Gamma(a_1)\,\Gamma(a_2) \cdots \Gamma(a_n)}\,. \tag{A.7}$$

In the case of two propagators with power one, this simplifies to

$$\frac{1}{A\,B} = \int_0^1 \mathrm{d}x \frac{1}{\big[x\,A + (1-x)\,B\big]^2}\,. \tag{A.8}$$

A.1.3 Loop-Integral Simplifications

In loop-integrals, there are a few simplification formulae we can employ to simplify manual calculations. One of these is to express scalar products of the loop momentum ℓ with external momenta q_i as propagators. See, for instance, [47]. Assume we have a set of n propagators of the form

$$D_i = \frac{1}{(\ell + q_i)^2 - m_i^2}\,, \tag{A.9}$$

with

$$D_n = \frac{1}{\ell^2 - m_n^2}\,. \tag{A.10}$$

We can then write for any scalar product $\ell \cdot q_i$,

$$\ell \cdot q_i = \frac{1}{2}\Big[\big((\ell + q_i)^2 - m_i^2\big) - \big(\ell^2 - m_n^2\big) - \big(q_i^2 - m_i^2\big) - m_n^2\Big] \tag{A.11}$$

$$= \frac{1}{2}\Big[D_i - D_n - q_i^2 + m_i^2 - m_n^2\Big]\,. \tag{A.12}$$

This formula allows us to reduce tensor integrals to a linear combination of lower-point, lower-rank tensor integrals, and scalar integrals.

Another useful relation for this purpose is

$$\ell^\mu\, \ell^\nu \longrightarrow \frac{g^{\mu\nu}}{d}\,\ell\,, \tag{A.13}$$

which is to be understood in the sense that we can substitute the left-hand side by the right-hand side in the integral.

We also list the equation for the one-loop integral over a single propagator raised to some power:

$$A(n,\,\Delta) \equiv \tilde{\mu}^{4-d} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \, \frac{1}{(\ell^2 - \Delta)^n} = \frac{i}{16\pi^2} \, \frac{\Gamma\!\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{\Delta}{4\pi\,\tilde{\mu}^2} \right)^{\frac{d}{2}-2} (-\Delta)^{2-n} \,, \tag{A.14}$$

which is particularly powerful in combination with the previously listed formulae. Note that we use the $\overline{\text{MS}}$ rescaled renormalization scale

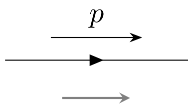
$$\tilde{\mu} = \mu \sqrt{\frac{e^{\gamma_E}}{4\pi}} \,. \tag{A.15}$$

A.2 Feynman-Rules

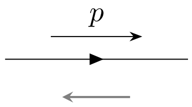
In this section, we list all the Feynman-rules for propagators and vertices used in this work.

A.2.1 Propagators

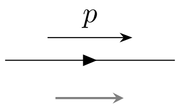
Here, we collect the propagators for all fields we used in calculations. Note that $\cancel{\text{SSB}}$ and SSB are used for vector bosons to indicate whether they are massive, i.e., whether Spontaneous Symmetry Breaking has taken place. Similarly, UG denotes the unitary gauge propagator. For non-gauge, massive vector bosons, the SSB propagator of the Z' can be used. Note that the indices for the vector bosons refer to group indices of the respective (non-abelian) gauge group. In the abelian case, these indices can be simply dropped.

l_a^f  l_b^g

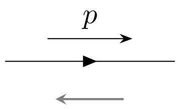
$$= \frac{i\,\not{p}}{p^2 + i\epsilon}\, \delta^{gf}\, \delta_{ba} \,, \tag{A.16}$$

l_a^f  l_b^g

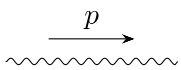
$$= \frac{-i\,\not{p}}{p^2 + i\epsilon}\, \delta^{gf}\, \delta_{ba} \,, \tag{A.17}$$

N_f  N_g

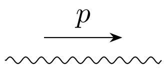
$$= \frac{i\,\not{p}}{p^2 + i\epsilon}\, \delta_{gf} \,, \tag{A.18}$$

N_f  N_g

$$= \frac{-i\,\not{p}}{p^2 + i\epsilon}\, \delta_{gf} \,, \tag{A.19}$$

Z'_μ  Z'_ν

$$\overset{\cancel{\text{SSB}}}{=} \frac{i}{p^2 + i\epsilon} \left[-\, g^{\mu\nu} + \big(1 - \tilde{\xi}\big)\, \frac{p^\mu\, p^\nu}{p^2} \right], \tag{A.20}$$

$V_{\mu,\,i}$  $V_{\nu,\,j}$

$$\overset{\cancel{\text{SSB}}}{=} \frac{i\,\delta_{ij}}{p^2 + i\epsilon} \left[-\, g^{\mu\nu} + \big(1 - \xi_n\big)\, \frac{p^\mu\, p^\nu}{p^2} \right], \tag{A.21}$$

$$V^{+}_{\mu,i} \xrightarrow{p} V^{+}_{\nu,j} \overset{\cancel{\text{SSB}}}{=} \frac{i\,\delta_{ij}}{p^2 + i\epsilon}\left[-g^{\mu\nu} + \left(1-\xi_c\right)\frac{p^\mu\, p^\nu}{p^2}\right], \tag{A.22}$$

$$V^{-}_{\mu,i} \xrightarrow{p} V^{-}_{\nu,j} \overset{\cancel{\text{SSB}}}{=} \frac{i\,\delta_{ij}}{p^2 + i\epsilon}\left[-g^{\mu\nu} + \left(1-\xi_c\right)\frac{p^\mu\, p^\nu}{p^2}\right], \tag{A.23}$$

$$Z'_{\mu} \xrightarrow{p} Z'_{\nu} \overset{\text{SSB}}{=} \frac{i}{p^2 - M^2_{Z'} + i\epsilon}\left[-g^{\mu\nu} + \left(1-\tilde{\xi}\right)\frac{p^\mu\, p^\nu}{p^2 - \tilde{\xi}\, M^2_{Z'}}\right], \tag{A.24}$$

$$V_{\mu,i} \xrightarrow{p} V_{\nu,j} \overset{\text{SSB}}{=} \frac{i\,\delta_{ij}}{p^2 - M^2_n + i\epsilon}\left[-g^{\mu\nu} + \left(1-\xi_n\right)\frac{p^\mu\, p^\nu}{p^2 - \xi_n\, M^2_n}\right], \tag{A.25}$$

$$V^{+}_{\mu,i} \xrightarrow{p} V^{+}_{\nu,j} \overset{\text{SSB}}{=} \frac{i\,\delta_{ij}}{p^2 - M^2_c + i\epsilon}\left[-g^{\mu\nu} + \left(1-\xi_c\right)\frac{p^\mu\, p^\nu}{p^2 - \xi_c\, M^2_c}\right], \tag{A.26}$$

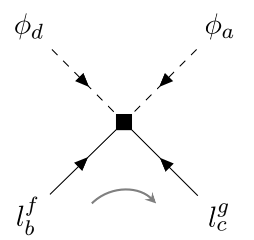
$$V^{-}_{\mu,i} \xrightarrow{p} V^{-}_{\nu,j} \overset{\text{SSB}}{=} \frac{i\,\delta_{ij}}{p^2 - M^2_c + i\epsilon}\left[-g^{\mu\nu} + \left(1-\xi_c\right)\frac{p^\mu\, p^\nu}{p^2 - \xi_c\, M^2_c}\right], \tag{A.27}$$

$$Z'_{\mu} \xrightarrow{p} Z'_{\nu} \overset{\text{UG}}{=} \frac{i}{p^2 - M^2_{Z'} + i\epsilon}\left[-g^{\mu\nu} + \frac{p^\mu\, p^\nu}{M^2_{Z'}}\right], \tag{A.28}$$

$$S_{ij} \xrightarrow{p} S_{kl} = \frac{i}{p^2 - m^2_{ij} + i\epsilon}\,\delta_{ij,\,kl}\,. \tag{A.29}$$

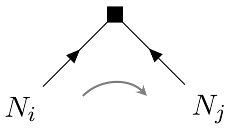
A.2.2 Vertices

Here, we collectively present the Feynman-rules for all the vertices we used in calculations. The T^i_{gf} are representatively charge matrices or generators of the respective (non-abelian) gauge group. For the scalar vertex with the Z' we use the convention of incoming momenta, $\partial_\mu \longrightarrow -i\,p_\mu$.

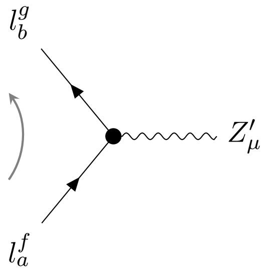


$$= i\,\tilde{\mu}^{(4-d)}\kappa_{gf}\,\underbrace{\frac{1}{2}\left(\varepsilon_{cd}\,\varepsilon_{ba} + \varepsilon_{ca}\,\varepsilon_{bd}\right)}_{\equiv E_{abcd}}\,P_L \tag{A.30}$$

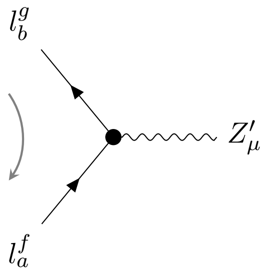
$$= i\,\tilde{\mu}^{(4-d)}\,\kappa_{gf}\,E_{abcd}\,P_L\,,$$



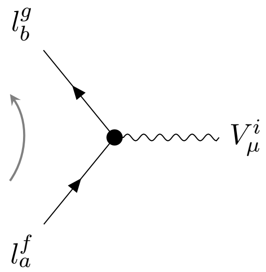
$$= \; -i\, M_{ji}\, P_R\,, \tag{A.31}$$



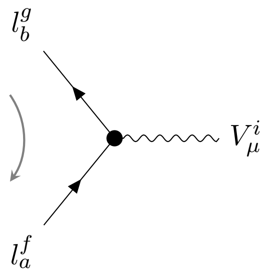
$$= \; -i\,\tilde{\mu}^{(4-d)/2}\,\tilde{g}\,\tilde{Q}_{gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{A.32}$$



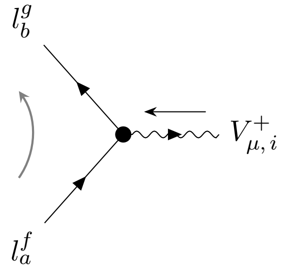
$$= \; -i\,\tilde{\mu}^{(4-d)/2}\,\tilde{g}\,\tilde{Q}_{gf}\,\big(-\gamma_\mu\,P_R\big)\,\delta_{ba}\,, \tag{A.33}$$



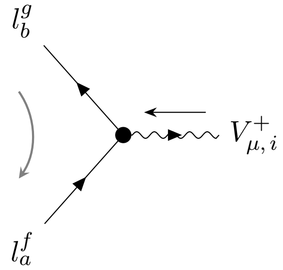
$$= \; -i\,\tilde{\mu}^{(4-d)/2}\,g_n\,T^i_{gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{A.34}$$



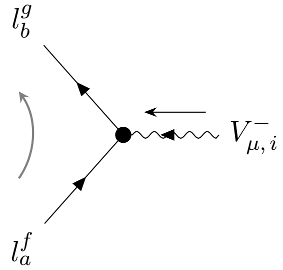
$$= \; -i\,\tilde{\mu}^{(4-d)/2}\,g_n\,T^i_{gf}\,\big(-\gamma_\mu\,P_R\big)\,\delta_{ba}\,, \tag{A.35}$$



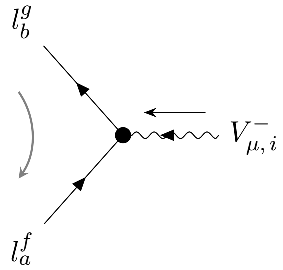
$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T^i_{+,\,gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{A.36}$$



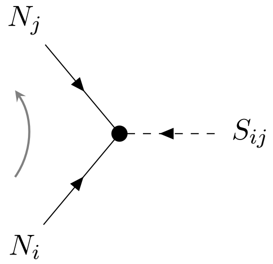
$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T^i_{+,\,gf}\,\big(-\gamma_\mu\,P_R\big)\,\delta_{ba}\,, \tag{A.37}$$



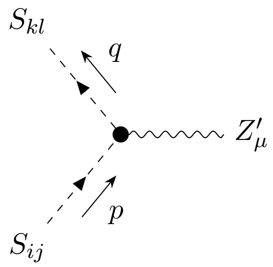
$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T^i_{-,\,gf}\,\gamma_\mu\,P_L\,\delta_{ba}\,, \tag{A.38}$$



$$= -\frac{i}{\sqrt{2}}\,\tilde{\mu}^{(4-d)/2}\,g_c\,T^i_{-,\,gf}\,\big(-\gamma_\mu\,P_L\big)\,\delta_{ba}\,, \tag{A.39}$$



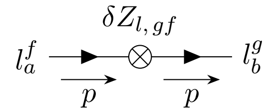
$$= -i\,\tilde{\mu}^{(4-d)/2}\,\lambda_{ji}\,P_R\,, \tag{A.40}$$



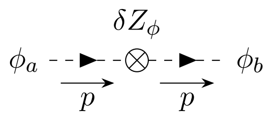
$$= -i\,\tilde{\mu}^{(4-d)/2}\,\tilde{g}\,Y^{S}_{ij}\,\delta_{ij,\,kl}\big(p_\mu + q_\mu\big)\,. \tag{A.41}$$

$$\tag{A.42}$$

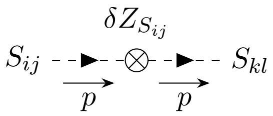
The relevant counterterm vertices are given by



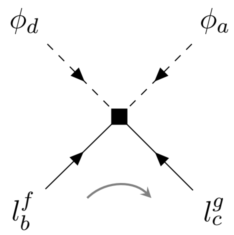
$$= i\,\not{p}\,\delta Z_{l,\,gf}\,P_L\,\delta_{ba}\,, \tag{A.43}$$



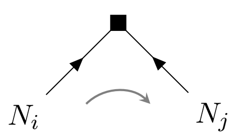
$$= i\,p^2\,\delta Z_\phi\,\delta_{ba}\,, \tag{A.44}$$



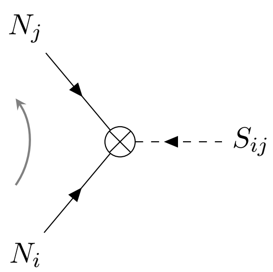
$$= i\,p^2\,\delta Z_{S_{ij}}\,\delta_{ij,\,kl}\,, \tag{A.45}$$



$$= i\,\tilde{\mu}^{(4-d)}\,\delta\kappa_{gf}\,E_{abcd}\,P_L\,, \tag{A.46}$$



$$= -i\,\delta M_{ji}\,P_R\,, \tag{A.47}$$



$$= -i\,\tilde{\mu}^{(4-d)/2}\,\delta\lambda_{ji}\,P_R\,. \tag{A.48}$$

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